AUXILIARY EQUATIONS FOR SOLVING NONLINEAR EVOLUTIONARY EQUATIONS

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Outline

Introduction

- Direct finding of travelling wave solutions for PDE's;
- How does the auxiliary equation influence the form of the solutions for the investigated equation?
- We use **Korteweg de Vries equation** as a "toy" model for proving how the functional expansion technique applies;
- We investigate **Benjamin-Bona-Mahony equation** and find interesting classes of solutions for different versions of auxiliary equations.

Travelling waves and the auxiliary equation technique

- **The travelling waves** represent an important class of solutions for mathematical models in the field of nonlinear dynamics. For a two dimensional model, defined by x and t, the wave variable is:

$$\xi = x - Vt, \quad V = wave\ velocity \tag{1}$$

- **The algorithm for finding travelling waves**:
(i) reduction of the PDE to an ODE;
(ii) choice of an adequate auxiliary equation, a supplementary equation with known solutions;
(iii) expressing the solutions for the investigated ODE in terms of those of the auxiliary equation.

Travelling waves and the auxiliary equation technique

- Let us consider the dependent variable $u(x,t)$ defined in a $2D$ space (x,t) satisfying the PDE:

$$F(u, u_t, u_x, u_{xx}, u_{tt}, ..) = 0 \tag{2}$$

The wave variable $\xi = x - Vt$ transforms (2) into an ODE:

$$\Delta(u, u', u'', \cdots) = 0; \quad u' = du(\xi)/d\xi \tag{3}$$

- For solving (3), we consider that its solutions could be expressed in terms of the known solutions $G(\xi)$ of an auxiliary equation of the form:

$$\Theta(G, G', G'', \cdots) = 0 \tag{4}$$

Travelling waves and the auxiliary equation technique

Examples of auxiliary equations usually met in literature:

$$G' = q_0 + q_1 G + q_2 G^2.$$

$$G' = \frac{A}{G} + BG + CG^3.$$

$$G' = c_2 G^2 + c_4 G^4 + c_6 G^6.$$

$$G'' + \lambda G' + \mu G = 0.$$

$$AGG'' + B(G')^2 + CGG' + EG^2 = 0.$$

Travelling waves and the auxiliary equation technique

- Usually, we express $u(\xi)$ in terms of $G(\xi)$ in the form:

$$u(\xi) = \sum_{i=-m}^{m} P_i(G) H^i(G, G', G'', \cdots). \tag{5}$$

  where $P_i(G)$ are $2m+1$ functionals depending on $G(\xi)$ and that have to be determinate.

- The value of the parameter m depends on the model and it is established through a balancing procedure among the terms of higher order derivative, respectively of the higher nonlinearity.
- The specific form of the generalized representations (5) also depends on the choice of the auxiliary equation: $\Theta(G, G', G'', \cdots) = 0$.

Higher order PDE's with soliton solutions

- **The KdV Equation** (introduced by Boussinesq in 1877 and rediscovered by Diederik Korteweg and G. de Vries in 1895):

$$u_t + u_{xxx} + 6uu_x = 0,$$

the first equation describing travelling waves in shallow water.

- **The Regularized Long-Wave (RLW) Equation**, also known as **Benjamin-Bona-Mahony (BBM) Equation**:

$$u_t + u_x + uu_x - u_{xt} = 0 \tag{6}$$

Describes the small-amplitude long wave of water in a channel.

Higher order PDE's with soliton solutions

- **The Camassa-Holm equation** was first proposed by Camassa and Holm for modelling the unidirectional propagation of irrotational water wave over a planar wall:

$$u_t + 2ku_x - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0$$

Here k is a constant related to gravity and initial undisturbed water depth.

- **The Rosenau equation** proposed for describing the wave-wave and wave-wall interactions that KdV equation is not working:

$$u_t + u_x + uu_x + u_{xxxt} = 0 \qquad (7)$$

Higher order PDE's with soliton solutions

- **The Rosenau-RLW equation** includes the term $-u_{xxt}$ from RLW, in the Rosenau Equation:

$$u_t - u_{xxt} + u_{xxxxt} + u_x + uu_x = 0, \qquad (8)$$

further extended into **the generalized Rosenau-RLW eq.**:

$$u_t - u_{xxt} + u_{xxxxt} + u_x + u^m u_x = 0, \quad (m \geq 1,\ \textit{positive integer})$$

- **The Rosenau-KdV equation** describes better the behaviour of nonlinear waves, by including the viscous term u_{xxx} in the Rosenau equation (7):

$$u_t + u_{xxxt} + u_{xxx} + u_x + uu_x = 0 \qquad (9)$$

Higher order PDE's with soliton solutions

- **The Rosenau-KdV-RLW equation** - by coupling the above Rosenau-RLW Equation and Rosenau-KdV Equation:

$$u_t - u_{xxt} + u_{xxxxt} + u_{xxx} + u_x + uu_x = 0$$

- **The Kawahara equation** arose in the theory of shallow water waves with surface tension:

$$u_t + u_x + uu_x + u_x xx - u_{xxxxx} = 0$$

- **The Rosenau-Kawahara equation** - by adding another viscous term$-u_{xxxx}$ to the Rosenau-KdV equation, (9):

$$u_t + u_x + uu_x + u_{xxx} + u_{xxxt} - u_{xxxxx} = 0$$

Higher order PDE's with soliton solutions

- **The generalized Rosenau-Kawahara Equation**:

$$u_t + au_x + b^m u_x + cu_{xxx} + \lambda u_{xxxxt} - vu_{xxxxx} = 0 \qquad (10)$$

Here a, b, c, λ, v are real constants; m is a positive integer, which indicates the power law nonlinearity.

- **The generalized Rosenau-Kawahara-RLW equation**:

$$u_t + au_x + bu^m u_x + cu_{xxx} - \alpha u_{xxt} + \lambda u_{xxxxt} - vu_{xxxxx} = 0$$

It is obtained by coupling the generalized Rosenau-RLW equation (8) and the generalized Rosenau-Kawahara equation (10).

The functional expansion method

- The functional expansion method, proposed in [5], supposes to have:

$$u(\xi) = \sum_{i=0}^{M} P_i(G)(G')^i = P_0 + P_1 G' + \ldots + P_M (G')^M \quad (11)$$

where $G' \equiv \frac{dG}{d\xi}$.

- The functionals $P_i(G), \quad i = 1, \ldots, M$, can be determined by solving the determining system generated when (11) is introduced in the ODE:

$$\Delta(u, u_\xi, \cdots) = 0.$$

Auxiliary equations and traveling waves for KdV

- We consider the KdV Equation:

$$u_t + uu_x + \delta u_{xxx} = 0$$

- Introducing the wave variable $\xi = x - Vt$ we get the ODE:

$$-VU + \frac{U^2}{2} + \delta U_{\xi\xi} = 0$$

- As auxiliary equation we will use:

$$G'' + \lambda G' + \mu G = 0$$

- The solution we are looking for has the form:

$$u(\xi) = \sum_{i=0}^{2} P_i(G)(G')^i = P_0 + P_1 G' + P_2 (G')^2 \qquad (12)$$

- Let us denote: $\dot{P}_i = \frac{dP_i}{dG}, i = 0, 1, 2$

Auxiliary equations and travelling waves for KdV

- By equating to zero the terms of the different powers of the derivatives $G(\xi)$, we get the following determining system of ODE's for the functionals (P_0, P_1, P_2):

$$\ddot{P}_0 = 3\lambda P_4 + 5\mu G P_5 - 2(2\lambda^2 - \mu)P_2 - \frac{1}{2\delta}P_1^2 - \frac{1}{\delta}P_0 P_2 + \frac{1}{\delta}VP_2$$

$$\ddot{P}_1 = 5P_5 - \frac{1}{\delta}P_1 P_2$$

$$\ddot{P}_2 = -\frac{1}{2\delta}P_2^2$$

$$-\delta\lambda P_3 - 3\delta\mu G P_4 + \delta(\lambda^2 - \mu)P_1 + 6\lambda\mu G P_2 + P_0 P_1 - VP_1 = 0$$

$$-\delta\mu P_3 + \frac{1}{2}P_0^2 - VP_0 + \delta\lambda\mu G P_1 + 2\delta\mu^2 G^2 P_2 + k = 0; \quad k = const.$$

Auxiliary equations and travelling waves for KdV

The known solutions for the auxiliary $G'' + \lambda G' + \mu G = 0$ are:
(i) if $\Delta = \lambda^2 - 4\mu > 0$ the auxiliary equation has a hyperbolic solution:

$$G(\xi) = e^{-(\lambda/2)\xi}\left(C_1 ch\frac{\sqrt{\Delta}}{2}\xi + C_2 sh\frac{\sqrt{\Delta}}{2}\xi\right).$$

(ii) if $\Delta = \lambda^2 - 4\mu < 0$ the solution will be expressed through periodic functions:

$$G(\xi) = e^{-(\lambda/2)\xi}\left(C_1 \cos\frac{\sqrt{-\Delta}}{2}\xi + C_2 \sin\frac{\sqrt{-\Delta}}{2}\xi\right).$$

(iii) if $\Delta = \lambda^2 - 4\mu = 0$ the solution will contain real exponentials:

$$G(\xi) = e^{-(\lambda/2)\xi}\left(C_1 + C_2\xi\right).$$

In the previous relations C_1 and C_2 denote arbitrary constants.

Auxiliary equations and travelling waves for KdV

Following the three cases, the solutions $u(\xi)$ is:

$$(i) \qquad u(\xi) = P_0 + 6\delta\lambda^2 - 6\delta\lambda\sqrt{\Delta}\frac{C_1 sh\frac{\sqrt{\Delta}}{2}\xi + C_2 ch\frac{\sqrt{\Delta}}{2}\xi}{C_1 ch\frac{\sqrt{\Delta}}{2}\xi + C_2 sh\frac{\sqrt{\Delta}}{2}\xi} - $$
$$-12\delta\left(-\frac{\lambda}{2} + \frac{\sqrt{\Delta}}{2}\frac{C_1 sh\frac{\sqrt{\Delta}}{2}\xi + C_2 ch\frac{\sqrt{\Delta}}{2}\xi}{C_1 ch\frac{\sqrt{\Delta}}{2}\xi + C_2 sh\frac{\sqrt{\Delta}}{2}\xi}\right)^2$$

$$(ii) \qquad u(\xi) = P_0 + 6\delta\lambda^2 - 6\delta\lambda\sqrt{-\Delta}\frac{-C_1 \sin\frac{\sqrt{-\Delta}}{2}\xi + C_2 \cos\frac{\sqrt{-\Delta}}{2}\xi}{C_1 \cos\frac{\sqrt{-\Delta}}{2}\xi + C_2 \sin\frac{\sqrt{-\Delta}}{2}\xi} - $$
$$-12\delta\left(-\frac{\lambda}{2} + \frac{\sqrt{-\Delta}}{2}\frac{-C_1 \sin\frac{\sqrt{-\Delta}}{2}\xi + C_2 \cos\frac{\sqrt{-\Delta}}{2}\xi}{C_1 \cos\frac{\sqrt{-\Delta}}{2}\xi + C_2 \sin\frac{\sqrt{-\Delta}}{2}\xi}\right)^2$$

$$(iii) \qquad u(\xi) = P_0 + 6\delta\lambda\lambda - 12\delta\lambda\frac{C_2}{(C_1 + C_2\xi)} - 12\delta\left(-\frac{\lambda}{2} + \frac{C_2}{(C_1 + C_2\xi)}\right)^2$$

Auxiliary equations and travelling waves for BBM

The equation under consideration is:

$$u_t + u_x + uu_x - u_{xxt} = 0. \tag{13}$$

With the wave variable $\xi = x - Vt$, BBM becomes the ODE:

$$2Vu_{\xi\xi} + u^2 + 2(1 - V)u + 2C = 0 \tag{14}$$

where C=constant of integration and V=velocity (real constant).
Let us consider the following auxiliary equation:

$$GG_{\xi\xi} - BG_\xi^2 - FGG_\xi - HG^2 = 0. \tag{15}$$

where B, F, H=const. coefficients. Solutions for (15) are known.

Auxiliary equations and travelling waves for BBM

For any solution $G(\xi)$ of the auxiliary equation (15), we will consider that the solution of (14) can be represented as:

$$u(\xi) = P_0(G) + P_1(G)G_\xi + P_2(G)(G_\xi)^2 \tag{16}$$

where $P_i(G), i = 0, 1, 2$ are arbitrary functionals that have to be determined.

Remark: In all the cases, taking into account that $\xi = x - Vt$, we can convert the mentioned solutions $u(\xi)$, coming back to the initial variables, x and t. We get solutions $u(x, t)$ for the BBM equation (13).

Auxiliary equations and travelling waves for BBM

Collecting G, G', G'^2, G'^3, G'^4 and putting the coefficients to zero, we get from (14) the determining system, a system of ODEs for the functionals $P_i, P_i' = \frac{dP_i}{dG}, i = 0, 1, 2$:

$$\begin{aligned}
&2C + 2P_0 - 2VP_0 + P_0^2 + 2FHVGP_1 + 4H^2VG^2P_2 + 2HVGP_0{}' = 0\\
&-P_1 + VP_1 - F^2VP_1 - HVP_1 - 2BHVP_1 - P_0P_1 - 6FHVGP_2 - FVP_0'\\
&\qquad - 3HVGP_1' = 0\\
&-6BFVP_1 - GP_1^2 - 2GP_2 + 2VGP_2 - 8F^2VGP_2 - 4HVGP_2 - 16BHVGP_2 -\\
&\qquad - 2GP_0P_2 - 2BVP_0' - 6FVGP_1' - 10HVG^2P_2' - 2VGP_0'' = 0\\
&-BVP_1 + 2B^2VP_1 + 10BFVGP_2 + G^2P_1P_2 + 3BVGP1' +\\
&\qquad + 5FVG^2P_2' + VG^2P_1'' = 0\\
&-4BVP_2 + 12B^2VP_2 + G^2P_2^2 + 10BVGP_2' + 2VG^2P_2'' = 0 \qquad (17)
\end{aligned}$$

Auxiliary equations and travelling waves for BBM

Remark: We were not able to analyze the system (17) in the most general case. In our investigation we considered the functionals P_0, P_1, P_2 having the forms of rational polynomials. Using a balancing procedure, we took:

$$\begin{aligned} P_0 &= \frac{\beta_0 + \beta_1 G + \beta_2 G^2}{\alpha_0 + \alpha_1 G + \alpha_2 G^2} \\ P_1 &= \frac{\gamma_0 + \gamma_1 G}{\alpha_0 + \alpha_1 G + \alpha_2 G^2} \\ P_2 &= \frac{\tau_0}{\alpha_0 + \alpha_1 G + \alpha_2 G^2} \end{aligned} \tag{18}$$

With these choices we get interesting new solutions for several cases.

Auxiliary equations and travelling waves for BBM

Case I. $B = 0, F \neq 0, H = 0,\ (F^4 - 1)V^2 + 2V + 2C - 1 = 0.$
The auxiliary equation (15) becomes:

$$G_{\xi\xi} - FG_{\xi} = 0 \tag{19}$$

and the functionals have the form:

$$P_0 = V - 1 - F^2 V, \quad P_1 = \frac{24FV\alpha_2}{\alpha_1 + 2\alpha_2 G}, \quad P_2 = -\frac{48V\alpha_2^2}{(\alpha_1 + 2\alpha_2 G)^2}.$$

I.1 For $F^2 > 0$, the auxiliary equation (19) admits the solution:

$$G(\xi) = C_2 + C_1 e^{F\xi} \tag{20}$$

The solution of (14) is:

$$u(\xi) = V - 1 - F^2 V + \frac{24C_1F^2V\alpha_2 e^{F\xi}}{\alpha_1 + 2\alpha_2(C_2 + C_1e^{F\xi})} - \frac{48C_1^2F^2V\alpha_2^2e^{2F\xi}}{[\alpha_1 + 2\alpha_2(C_2 + C_1e^{F\xi})]^2} \tag{21}$$

Auxiliary equations and travelling waves for BBM

I.2 For $F^2 < 0$, the auxiliary equation (19), admits the solution:

$$G(\xi) = e^{F\xi/2}(C_1 \sin \Delta\xi/2 + C_2 \cos \Delta\xi/2), \quad \Delta^2 = -F^2 > 0. \tag{22}$$

In this case the solution of (14) is:

$$\begin{aligned} u(\xi) &= V - 1 - F^2 V + \\ &+ \frac{12FV\alpha_2 e^{F\xi/2}[(FC_1 - \Delta C_2)\sin \Delta\xi/2 + (FC_2 + \Delta C_1)\cos \Delta\xi/2]}{\alpha_1 + \alpha_2 e^{F\xi/2}(C_1 \sin \Delta\xi/2 + C_2 \cos \Delta\xi/2)} - \\ &- \frac{12V\alpha_2 e^{F\xi}[(FC_1 - \Delta C_2)\sin \Delta\xi/2 + (FC_2 + \Delta C_1)\cos \Delta\xi/2]^2}{\alpha_1 + \alpha_2 e^{F\xi/2}(C_1 \sin \Delta\xi/2 + C_2 \cos \Delta\xi/2)^2} \end{aligned} \tag{23}$$

Auxiliary equations and travelling waves for BBM

Case II. $B \neq 0, F \neq 0, H \neq 0$ and the constraint
$2C + 2V + (-1 + F^4 - 8(-1+B)F^2H + 16(-1+B)^2H^2)V^2 - 1 = 0.$
The auxiliary equation (15) takes its most general form

$$G_{\xi\xi}G - BG_\xi^2 - FGG_\xi - HG^2 = 0$$

and the functionals are:

$$P_0 = V - 1 - F^2V + 8HV(1-B), \quad P_1 = \frac{12FV(1-B)}{G}, \quad P_2 = -\frac{12V(1-B)^2}{G^2}$$

II.1 For $F^2 + 4H(1-B) > 0$, the auxiliary equation (15) admits the solution:

$$G(\xi) = [e^{F\xi/2}(C_1 e^{\sqrt{F^2+4H(1-B)}\xi/2} + C_2 e^{-\sqrt{F^2+4H(1-B)}\xi/2}]^{\frac{1}{1-B}} \qquad (24)$$

We get a solution of (14) in the form:

$$\begin{aligned} u(\xi) &= V - 1 - F^2V - 4HV(1-B) + \\ &+ \frac{12C_2V(F^2+4H(1-B)}{C_2 + C_1 e^{\sqrt{F^2+4H(1-B)}\xi}} - \frac{12C_2^2V(F^2+4H(1-B)}{[C_2 + C_1 e^{\sqrt{F^2+4H(1-B)}\xi}]^2} \end{aligned} \qquad (25)$$

Auxiliary equations and travelling waves for BBM

II.2 For $F^2 + 4H(1-B) < 0$, the auxiliary equation (15) admits the solution:

$$G(\xi) = [e^{\frac{F\xi}{2}}(C_1 \sin\sqrt{-F^2 - 4H(1-B)}\frac{\xi}{2} + C_2 \cos\sqrt{-F^2 - 4H(1-B)}\frac{\xi}{2})]^{\frac{1}{1-B}}. \tag{26}$$

In this case the solution of (14) is, after simplification:

$$u(\xi) = V - 1 - F^2V - 4HV(1-B) + \\ + \frac{3V(C_1^2 + C_2^2)(F^2 + 4H(1-B))}{[C_1 \sin\sqrt{-F^2 - 4H(1-B)}\xi/2 + C_2 \cos\sqrt{-F^2 - 4H(1-B)}\xi/2]^2} \tag{27}$$

Conclusions

- The paper pointed out travelling wave solutions of some non-linear PDE's.
- The auxiliary equation method and the functional expansion technique were used and the main aim was to investigate how the travelling waves depend on the form of the auxiliary equation.
- The conclusion dropped out by studying the Benjamin-Bona-Mahony equation is that, usually, the travelling waves inherit important characteristics of the solutions accepted by the auxiliary equation. It was possible to get hyperbolic or periodic solutions of BBM starting from similar solutions of the considered auxiliary equations.

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