On p-Adic Matter

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1. Introduction

I plan to

- give a brief review of basic properties of p-adic strings
- show how a matter can be derived from p-adic strings
- show that this p-adic matter is related to evolution of a closed universe
- discuss obtained results.

$$
\begin{tikzcd}
 & & & \mathbb{R} \arrow[r, dash] & \mathbb{C} \\
\mathbb{N} \arrow[r, dash] & \mathbb{Z} \arrow[r, dash] & \mathbb{Q} \arrow[ur, dash, "|\cdot|_\infty"] \arrow[dr, dash, "|\cdot|_p"'] & & \\
 & & & \mathbb{Q}_p \arrow[r, dash] & \mathbb{C}_p
\end{tikzcd}
$$

Figure: Real and *p*-adic numbers

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2. p-Adic Strings

Volovich, Vladimirov, Freund, Witten, Arefeva, BD, ...
String amplitudes:

- standard crossing symmetric Veneziano amplitude

$$
\begin{aligned}
A_\infty(a,b) &= g_\infty^2 \int_{\mathbb{R}} |x|_\infty^{a-1}\, |1-x|_\infty^{b-1}\, d_\infty x \\
&= g_\infty^2 \frac{\zeta(1-a)}{\zeta(a)} \frac{\zeta(1-b)}{\zeta(b)} \frac{\zeta(1-c)}{\zeta(c)}
\end{aligned}
$$

- p-adic crossing symmetric Veneziano amplitude

$$
\begin{aligned}
A_p(a,b) &= g_p^2 \int_{\mathbb{Q}_p} |x|_p^{a-1}\, |1-x|_p^{b-1}\, d_p x \\
&= g_p^2 \frac{1-p^{a-1}}{1-p^{-a}} \frac{1-p^{b-1}}{1-p^{-b}} \frac{1-p^{c-1}}{1-p^{-c}}
\end{aligned}
$$

where $a = -s/2 - 1$ and $a, b, c \in \mathbb{C}$ and $a + b + c = 1$.

2. p-Adic Strings

1. Euler product formula for Riemann zeta function

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}, \quad \Re s > 1$$

2. Freund-Witten product formula for adelic strings

$$A(a,b) = A_\infty(a,b) \prod_p A_p(a,b) = g_\infty^2 \prod_p g_p^2 = const.$$

- amplitudes on equal footing
- various faces of an adelic string
- amplitude of ordinary strings can be regarded as product of p-adic inverses

3. Effective Field Theory for p-Adic Strings

- The exact tree-level Lagrangian for effective scalar field φ which describes open p-adic string (tachyon) amplitudes

$$\mathcal{L}_p = \frac{m_p^D}{g_p^2} \frac{p^2}{p-1} \Big[-\frac{1}{2} \varphi\, p^{-\frac{\Box}{2m_p^2}} \varphi + \frac{1}{p+1} \varphi^{p+1} \Big]$$

p is any prime number, $\Box = -\partial_t^2 + \nabla^2$ D-dimensional d'Alembertian and metric with signature $(- + ... +)$ (Freund, Witten, Frampton, Okada, ...) .

3. Effective Field Theory for p-Adic Strings

The above Lagrangian is written completely in terms of real numbers and there is no explicit dependence on the p-adic world sheet. However, it can be rewritten as:

$$\mathcal{L}_p = \frac{m^D}{g^2}\,\frac{p^2}{p-1}\Big[\frac{1}{2}\varphi\int_{\mathbb{R}}\Big(\int_{\mathbb{Q}_p\setminus\mathbb{Z}_p}\chi_p(u)|u|_p^{\frac{k^2}{2m^2}}\,du\Big)\tilde{\varphi}(k)\,\chi(kx)\,d^Dk + \frac{1}{p+1}\,\varphi^{p+1}\Big],$$

where $\chi(kx) = e^{-ikx}$. Since $\int_{\mathbb{Q}_p}\chi_p(u)|u|^{s-1}du = \frac{1-p^{s-1}}{1-p^{-s}} = \Gamma_p(s)$ and it is present in the scattering amplitude, one can say that variable u in $\int_{\mathbb{Q}_p\setminus\mathbb{Z}_p}\chi_p(u)|u|_p^{\frac{k^2}{2m^2}}du = -p^{\frac{k^2}{2m^2}}$ is related to the p-adic string world-sheet.

3. Effective Field Theory for p-Adic Strings

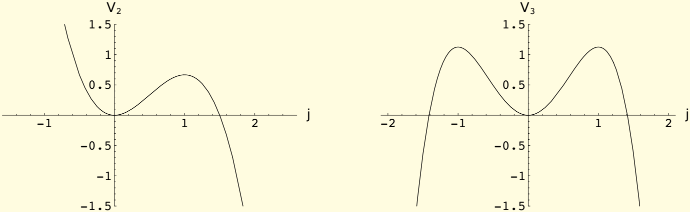


Figure: The 2-adic string potential $\mathcal{V}_2(\varphi)$ (on the left) and 3-adic potential $\mathcal{V}_3(\varphi)$ (on the right)

Potential .

$$\mathcal{V}_p(\varphi) = \frac{m_p^D}{g_p^2}\frac{p^2}{p-1}\left[\frac{1}{2}\varphi^2 - \varphi^{p+1}\right].$$

3. Effective Field Theory for p-Adic Strings

- The equation of motion is

$$p^{-\frac{\Box}{2m^2}}\,\varphi = \varphi^p, \quad \varphi = 0,\ \varphi = 1,\ (\varphi = -1,\ p \neq 2)$$

$$e^{A\partial_t^2}\, e^{Bt^2} = \frac{1}{\sqrt{1-4AB}}\, e^{\frac{Bt^2}{1-4AB}}, \quad 1 - 4AB > 0$$

There are also nontrivial solutions:.

$$\varphi(x^i) = p^{\frac{1}{2(p-1)}}\ \exp\Big(-\frac{p-1}{2m^2\, p \ln p}(x^i)^2\Big)$$

$$\varphi(t) = p^{\frac{1}{2(p-1)}}\ \exp\Big(\frac{p-1}{2p\ln p} m^2 t^2\Big)$$

$$\varphi(x) = p^{\frac{D}{2(p-1)}}\ \exp\Big(-\frac{p-1}{2\,p\ln p}\, m^2\, x^2\Big), \quad x^2 = -t^2 + \sum_{i=1}^{D-1} x_i^2 .$$

3. Effective Field Theory for p-Adic Strings

- When $p = 1 + \varepsilon \to 1$ there is the limit which is related to the ordinary bosonic string in the boundary string field theory (Gerasimov-Shatashvili):

$$\mathcal{L} = \frac{m^D}{g^2}\left[\frac{1}{2}\,\varphi\,\frac{\Box}{m^2}\,\varphi + \frac{\varphi^2}{2}\left(\ln\varphi^2 - 1\right)\right]$$

- From several considerations was shown that some nontrivial features of ordinary strings are similar to p-adic ones and are related to the p-adic effective action.
- If ordinary matter has its origin in ordinary string, then may be there is p-adic matter related to p-adic strings!

4. p-Adic Matter in Minkowski space

To avoid tachyon, consider transition $m^2 \to -m^2$ in D = 4 dimensions. Also change sign to Lagrangian to avoid ghost. Then the related new Lagrangian is

$$L_p = \frac{m^4}{g^2}\,\frac{p^2}{p-1}\Big[\frac{1}{2}\,\phi\,p^{\frac{\Box}{2m^2}}\,\phi - \frac{1}{p+1}\,\phi^{p+1}\Big] \qquad (1)$$

with the corresponding potential

$$V_p(\phi) = \frac{m^4}{g^2}\,\frac{p^2}{p-1}\Big[\frac{1}{p+1}\,\phi^{p+1} - \frac{1}{2}\,\phi^2\Big].$$

and equation of motion

$$p^{\frac{\Box}{2m^2}}\,\phi = \phi^p \qquad (2)$$

4. p-Adic Matter in Minkowski space

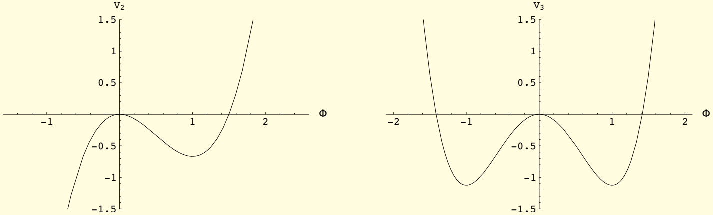


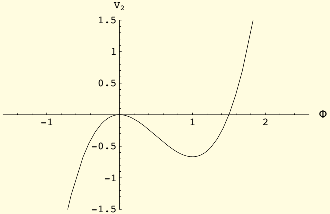
Figure: New potentials $V_2(\phi)$ and $V_3(\phi)$, which are related to new Lagrangian.

Trivial solutions

$$p^{\frac{\Box}{2m^2}}\,\phi = \phi^p, \quad \phi = 0,\ \phi = 1,\ (\phi = -1,\ p \neq 2)$$

and also previous nontrivial solutions with $m^2 \to -m^2$.

4. p-Adic Matter in Minkowski space



Consider weak field approximation $\phi = 1 + \theta, \quad |\theta| \ll 1$.

$$p^{\frac{\Box}{2m^2}} (1+\theta) = (1+\theta)^p, \quad \Rightarrow \quad p^{\frac{\Box}{2m^2}} \, \theta = p \, \theta .$$

EoM $p^{\frac{\Box}{2m^2}} \, \theta = p \, \theta$ has solution since the following Klein-Gordon equation $(\Box - 2m^2) \, \theta = 0$, is satisfied and $\theta \sim a \, e^{i(-Et + \vec{k}\vec{x})} + \bar{a} \, e^{-i(-Et+\vec{k}\vec{x})}$ is a scalar field with $E^2 = 2m^2 + \vec{k}^2$.

5. A Closed Universe with p-Adic Matter

A 4-dimensional gravity with a nonlocal scalar field ϕ and cosmological constant Λ, given by the EH action

$$S = \gamma \int d^4x \sqrt{-g}\,(R - 2\Lambda) + S_m,$$

where $\gamma = \frac{1}{16\pi G}$, R is the Ricci scalar and

$$S_m = \sigma \int d^4x \sqrt{-g}\,\Big(\frac{1}{2}\phi F(\Box)\phi - U(\phi)\Big),$$

where $F(\Box) = \sum_{n=0}^{\infty} f_n\, \Box^n$ and $U(\phi)$ is a part of the potential. Note that now

$$\Box = \nabla_\mu \nabla^\mu = \frac{1}{\sqrt{-g}}\partial_\mu \sqrt{-g} g^{\mu\nu}\partial_\nu$$

.

5. A Closed Universe with p-Adic Matter

The equations of motion for $g_{\mu\nu}$ and ϕ are

$$\gamma(G_{\mu\nu} + \Lambda g_{\mu\nu}) - \frac{\sigma}{4}\, g_{\mu\nu}\, \phi F(\Box)\phi + g_{\mu\nu}\, \frac{\sigma}{2}\, U(\phi) + \frac{\sigma}{4}\, \Omega_{\mu\nu}(\phi) = 0,$$
$$F(\Box)\phi - U'(\phi) = 0,$$

where

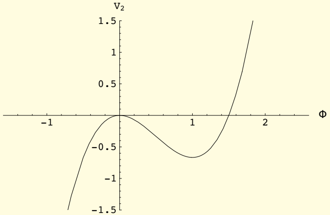
$$\Omega_{\mu\nu}(\phi) = \sum_{n=1}^{\infty} f_n \sum_{\ell=0}^{n-1} \Big[g_{\mu\nu}\, \big(\nabla^\alpha \Box^\ell \phi \nabla_\alpha \Box^{n-1-\ell} \phi + \Box^\ell \phi \Box^{n-\ell} \phi \big) - 2 \nabla_\mu \Box^\ell \phi \nabla_\nu \Box^{n-1-\ell} \phi \Big].$$

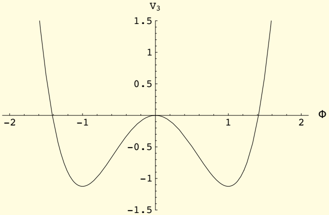
5. A Closed Universe with p-Adic Matter

Matter of interest is p-adic scalar field

$$S_p = \sigma_p \int d^4x \sqrt{-g} \left(\frac{1}{2} \phi \, p^{\frac{1}{2m^2}\Box} \, \phi - \frac{1}{p+1} \, \phi^{p+1} \right),$$

where $\sigma_p = \frac{m_p^D}{g_p^2} \frac{p^2}{p-1}$.





EoM for this p-adic field ϕ is $p^{\frac{1}{2m^2}\Box}\phi \equiv e^{\frac{\ln p}{2m^2}\Box}\phi = \phi^p$, It has the same trivial solutions as in the Minkowski space-time.

5. A Closed Universe with p-Adic Matter

We are interested in cosmological solutions of EoM in the homogeneous and isotropic space given by the FLRW metric

$$ds^2 = -dt^2 + a^2(t)\Big(\frac{dr^2}{1-kr^2} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2\Big),$$

where $a(t)$ is the cosmic scale factor, and $k = 0, +1, -1$ for the flat, closed and open universe, respectively. Owing to symmetries, there are only two independent EoM: trace

$$4\Lambda - R - \sigma\phi F(\Box)\phi + 2\sigma U(\phi) + \frac{\sigma}{4}\Omega = 0$$

and 00-component

$$\gamma(G_{00} - \Lambda) + \frac{\sigma}{4}\,\phi F(\Box)\phi - \frac{\sigma}{2}\,U(\phi) + \frac{\sigma}{4}\,\Omega_{00}(\phi) = 0,$$

where $\Omega = g^{\mu\nu}\Omega_{\mu\nu}$.

5. A Closed Universe with p-Adic Matter

We look for a solution of EoM in a weak field approximation $\phi = 1 + \theta$, where $|\theta| \ll 1$.

$$p^{\frac{\Box}{2m^2}}\,(1+\theta) = (1+\theta)^p, \quad \Rightarrow \quad p^{\frac{\Box}{2m^2}}\,\theta = p\,\theta,$$

where now

$$\Box = -\frac{\partial^2}{\partial t^2} - 3H\frac{\partial}{\partial t}, \quad H = \frac{\dot{a}}{a}.$$

$$p^{\frac{\Box}{2m^2}}\,\theta = p\,\theta$$

has solution if there is solution of $\Box\theta = 2m^2\theta$, i.e.

$$\frac{\partial^2\theta}{\partial t^2} + 3H\frac{\partial\theta}{\partial t} + 2m^2\theta = 0,$$

where $H = \dot{a}/a$ is the Hubble parameter.

5. A Closed Universe with p-Adic Matter

The simplest case is $H = constant$ and it corresponds to the scale factor $a(t) = Ae^{Ht}$. There is solution in the form $\theta(t) = C\ e^{\lambda t}$, where λ must satisfy quadratic equation

$$\lambda^2 + 3H\lambda + 2m^2 = 0.$$

Simple solutions $\lambda_{1,2} = \pm m$ and the general solution can be written as

$$\theta(t) = C_1\ e^{-mt} + C_2\ e^{mt} = \theta_1(t) + \theta_2(t),$$

where C_1 and C_2 are integration constants. Note that H and λ must have opposite sign. We have pairs:
$\theta_1(t) = C_1 e^{-mt}, \ \ a_1(t) = A_1 e^{mt}$ and
$\theta_2(t) = C_2 e^{mt}, \ \ a_2(t) = A_2 e^{-mt}$.

5. A Closed Universe with p-Adic Matter

The next step is to explore how solution for $\theta(t)$ satisfies EoM for gravitational field. The EH action with θ field is

$$S = \gamma \int d^4x \sqrt{-g}\,(R - 2\Lambda) + \sigma_p \int d^4x \sqrt{-g}\,\Big(\frac{1}{2}\,\theta p^{\frac{\Box}{2m^2}}\theta - \frac{p}{2}\theta^2 + \alpha_p\Big),$$

where $\alpha_p = \frac{p-1}{2(p+1)}$.
The potential $V_p(\theta) = -L_p(\Box = 0)$ is

$$V_p(\theta) = \sigma_p\big(\frac{p-1}{2}\theta^2 - \alpha_p\big) \tag{3}$$

and it has the form resembling that of the harmonic oscillator.

5. A Closed Universe with p-Adic Matter

With relevant replacements

$$\phi \to \theta, \quad \sigma \to \sigma_p, \quad U(\theta) = \frac{p}{2}\theta^2 - \alpha_p,$$

$$\gamma(4\Lambda - R) - \sigma_p\, \theta F(\Box)\theta + 2\sigma_p\, (\frac{p}{2}\, \theta^2 - \alpha_p) + \frac{\sigma_p}{4}\, \Omega = 0,$$
$$\gamma(G_{00} - \Lambda) + \frac{\sigma_p}{4}\, \theta F(\Box)\theta - \frac{\sigma_p}{2}\, (\frac{p}{2}\theta^2 - \alpha_p) + \frac{\sigma_p}{4}\, \Omega_{00}(\theta) = 0.$$

$$F(\Box) = p^{\frac{\Box}{2m^2}} = \sum_{n=0}^{\infty} \Big(\frac{\ln p}{2m^2}\Big)^n \frac{1}{n!}\, \Box^n = \sum_{n=0}^{\infty} f_n \Box^n .$$

Since $p^{\frac{\Box}{2m^2}}\, \theta = p\, \theta$, it simplifies the above equations

$$\gamma(4\Lambda - R) - 2\sigma_p\alpha_p + \frac{\sigma_p}{4}\, \Omega = 0,$$
$$\gamma(G_{00} - \Lambda) + \frac{\sigma_p}{2}\, \alpha_p + \frac{\sigma_p}{4}\, \Omega_{00}(\theta) = 0.$$

5. A Closed Universe with p-Adic Matter

Recall that in the FLRW metric

$$R_{00} = -3\frac{\ddot a}{a}, \quad G_{00} = 3\Big(\frac{\dot a^2}{a^2} + \frac{k}{a^2}\Big), \quad R = 6\Big(\frac{\ddot a}{a} + \frac{\dot a^2}{a^2} + \frac{k}{a^2}\Big),$$

computation for $a_1(t) = A_1 e^{mt}$ and $a_2(t) = A_2 e^{-mt}$ gives

$$\begin{aligned} R_{00}^{(1)} &= R_{00}^{(2)} = -3m^2, \\ G_{00}^{(1)} &= 3\Big(m^2 + \frac{k}{A_1^2}\, e^{-2mt}\Big), \quad G_{00}^{(2)} = 3\Big(m^2 + \frac{k}{A_2^2}\, e^{2mt}\Big), \\ R_1 &= 6\Big(2m^2 + \frac{k}{A_1^2}\, e^{-2mt}\Big), \quad R_2 = 6\Big(2m^2 + \frac{k}{A_2^2}\, e^{2mt}\Big). \end{aligned}$$

Direct calculation of $\Omega = g^{\mu\nu}\, \Omega_{\mu\nu}(\theta)$ and $\Omega_{00}(\theta)$ gives

$$\begin{aligned} \Omega_1(\theta) &= 3p \ln p\, \theta_1^2, \quad \Omega_2(\theta) = 3p \ln p\, \theta_2^2, \\ \Omega_{00}^{(1)} &= -\frac{3}{2} p \ln p\, \theta_1^2, \quad \Omega_{00}^{(2)} = -\frac{3}{2} p \ln p\, \theta_2^2. \end{aligned}$$

5. A Closed Universe with p-Adic Matter

One can easily verify that EoM are satisfied in both cases

$$\gamma(4\Lambda - R) - 2\sigma_p\alpha_p + \frac{\sigma_p}{4}\,\Omega = 0,$$
$$\gamma(G_{00} - \Lambda) + \frac{\sigma_p}{2}\,\alpha_p + \frac{\sigma_p}{4}\,\Omega_{00}(\theta) = 0,$$
$$p^{\frac{\Box}{2m^2}}\,\theta = p\,\theta$$

with conditions $6\gamma m^2 + \sigma_p\alpha_p - 2\gamma\Lambda = 0, \quad p\ln p\,\sigma_p\,A_i^2 C_i^2 - 8\gamma\,k = 0, \quad (i = 1, 2), \quad k = +1$, or in the more explicit form

$$\Lambda = 3m^2 + \frac{4\pi G}{g^2}\frac{p^2}{p-1}m^4, \qquad \frac{1}{(A_1C_1)^2} = \frac{1}{(A_2C_2)^2} = \frac{2\pi G}{g^2}\frac{p^3\ln p}{p-1}m^4.$$

6. Conclusion

- *p*-Adic strings are nonlocal, nonlinear and non-Archimedean objects with several ways related to ordinary strings.
- By slight modification of Lagrangian for *p*-adic strings follows scalar matter that makes sense.
- In a closed universe with *p*-adic matter and cosmological constant, there is exponential expansion (contraction)

$$\theta(t) = Ce^{\mp mt}, \qquad a(t) = Ae^{\pm mt}$$

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THANK YOU FOR YOUR ATTENTION!