About Geometrically Integrable Discrete Dynamical Systems and Their Periodic Orbits

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Content

§1. The Integrability Problem for Discrete Dynamical Systems.

§2. The Definition and Criteria for Geometric Integrability.

§3. Periodic Behavior of Integrable Maps and the Topological Entropy.

This talk is based on results of the paper

[1] L.E., "Geometrically integrable maps in the plane and their periodic orbits", Lobachevskii J. Math., **42**:10 (2021), 2315–2324.

Introduction

First works contained deep results on the integrability of discrete dynamical systems are works by G.Julia, P.Fatou and J.Ritt (although the term "integrability"is not used there). The problem considered in the above works is the description of pairs of commuting rational (in particular, polynomial) maps (in the projective plane)

$$H(G(x)) = G(H(x)).$$

The important conclusion follows from the papers of the above authors: the existence of the commuting maps with suitable properties implies integrability of the corresponding dynamical systems (in the sense of the explicit description of dynamics) (see [2] A.P. Veselov, "Integrable maps", Russian Math. Surveys, **48**:5(281) (1991), 3-42).
One can consider another function ψ instead G in the right part of the above equality. Then we obtain the integrability definition by R.I. Grigorchuk.

The General Definition

Birkhoff has written: "If we try to formulate the exact definition of integrability, we see that many definitions are possible, and every of them is of a specific theoretical interest."

Let Π be a compact curvilinear trapezoid in the plane $\mathbf{R^2}$ such that the section of Π by a straight line $y = constant$ (if it is not empty) be a nondegenerate closed interval.

Definition 1. We say that a self-map G of the curvilinear trapezoid Π is *geometrically integrable on G-invariant set* $A(G) \subseteq \Pi$ if there exist a self-map ψ of an interval J of the real line $\mathbf{R^1}$ and ψ-invariant set $B(\psi) \subseteq J$ such that the restriction $G_{|A(G)}$ is semiconjugate with the restriction $\psi_{|B(\psi)}$ by means of a continuous surjection $H : A(G) \to B(\psi)$, i.e. the following equality holds:

$$H \circ G_{|A(G)} = \psi_{|B(\psi)} \circ H. \tag{1}$$

The map $\psi_{|B(\psi)}$ is said to be *the quotient* of $G_{|A(G)}$.

[3] S. Belmesova and L. E., “On the concept of integrability for discrete dynamical systems. Investigation of wandering points of some trace map,” Nonlin. maps and their applic., Springer Proc. Math. Statist. **112**, Springer, Cham, 127–158 (2015).

[4] L.E., "Small C^1-smooth perturbations of skew products and the partial integrability property", Applied Math. and Nonlinear Sci. **5** (2), 317–328 (2020).

[5] L.E., "The Trace Map and Integrability of the Multifunctions,"J. Phys.: Conf. Ser., **990**, 012003 (2018).

[6] L.E., "Dynamics of skew products of interval maps", Russian Math. Surveys, 72:1 (2017), 101–178.

Examples of integrable maps, I

Example 1. Consider a skew product $F : J^2 \to J^2$, where J^2 is a closed rectangle in the plane, $J^2 = J \times J'$ (J, J' are closed interval), i.e. F is given by the formula

$$F(x, y) = (f(x), g_x(y)), \text{ where } g_x(y) = g(x, y), (x, y) \in J^2.$$

Example 2. Integrable maps arise in investigations of self-similar groups (see, e.g., [7] N-B. Dang, R. Grigorchuk, and M. Lyubich, "Self-similar groups and holomorphic dynamics: renormalization, integrability, and spectrum," arXiv:2010.00675v2, 74 p. (2021)).
Example 3. The trace map of the form

$$F(x, y) = (xy, (x-2)^2), \text{ where } (x, y) \in \mathbf{R}^2,$$

is integrable on the subset of the first quadrant that coincides with the exterior of the triangle $\Delta = \{(x, y) : x, y \geqslant 0; x + y \leqslant 4\}$ in this quadrant; F is topologically conjugate with Lotka-Volterra map $(x, y) \to (x(4 - x - y), xy)$.

Examples of integrable maps, II

Example 4. In recent papers (see [4] L.E., "Small C^1-smooth perturbations of skew products and the partial integrability property", Applied Math. and Nonlinear Sci. **5** (2), 317–328 (2020); [8] L.E., "Small perturbations of smooth skew products and Sharkovsky's theorem", Journ. Differ. Eq. and Applic. **26** (8), 1192–1211 (2020)) maps of the form

$$\Phi(x,\, y) = (f(x) + \mu(x,\, y),\, g_x(y)),\ \text{where}\ (x,\, y) \in J^2 \qquad (2)$$

were considered under the following conditions:
(i_Φ) maps (2) are C^1-smooth on J^2;
(ii_Φ) $\Phi(\partial J^2) \subset \partial J^2$, where $\partial(\cdot)$ is the boundary of a set;
(i_μ) the equality $\mu(x,\, y) = 0$ holds for every $(x,\, y) \in \partial J^2$;
(i_f) f is Ω-stable in the space of C^1-smooth self-maps of the interval J with the invariant boundary;
(ii_μ) the standard C^1-norm of μ satisfies some conditions of smallness that is connected with the previous condition (i_f).

Reference for the Definition of a Local Lamination

[9] D. Anosov and Ye. Zhuzhoma, “Nonlocal asymptotic behavior of curves and leaves of laminations on universal coverings,” (Russian) Tr. Mat. Inst. Steklova; translation in Proc. Steklov Inst. Math. **249** (2), 1–219 (2005).

The Definition of the Local One-Dimensional Lamination

Definition 2. Let A be a subset of Π satisfying $A = \bigcup\limits_{\alpha} L_\alpha$, where α belongs to an index set; curves L_α are pairwise disjoint. We say that the family of curves $\{L_\alpha\}$ has the local structure of a one-dimensional continuous lamination if for every point $x \in A$ there exist a neighborhood $U(x) \subset \Pi$ and a homeomorphism $\chi : U(x) \to \mathbf{R}^2$ such that every connected component of the intersection $U(x) \bigcap L_\alpha$ (if it is not empty) is mapping by means of χ into a straight line such that

$$\chi_{|U(x)\bigcap L_\alpha} : U(x) \bigcap L_\alpha \to \chi(U(x) \bigcap L_\alpha) \text{ is a homeomorphism.}$$

We say that the family $\{L_\alpha\}$ is a one-dimensional continuous *local lamination (without singularities)*. The set A is said to be *the support of the above local lamination* L, curves L_α are said to be *fibres*. If A is a closed set, $A \neq \Pi$, then we say about the one-dimensional continuous *lamination*; if $A = \Pi$ then we say about the one-dimensional continuous *foliation*.

The Geometric Criterion of the Integrability

Theorem 1. *Let* Π *be a compact trapezoid in the plane* $\mathbf{R^2}$*,* G *be a self-map of* Π*,* $A(G)$ *be a closed* G*-invariant subset of* Π *satisfying*

$$pr_2(A(G)) = pr_2(\Pi),\ pr_2 : \mathbf{R^2} \to Oy \text{ is the natural projection.} \quad (3)$$

Let J *be a segment of the line* $\mathbf{R^1}$*,* ψ *be a self-map of* J*,* $B(\psi)$ *be a closed* ψ*-invariant subset of* J*.*
Then $G_{|A(G)}$ *is the geometrically integrable map with the quotient* $\psi_{|B(\psi)}$ *by means of a continuous surjection* $H : A(G) \to B(\psi)$ *such that for every* $y \in pr_2(\Pi)$ *the map* H *is an injection on* x*, if and only if* $A(G)$ *is the support of a continuous invariant lamination for* $A(G) \neq \Pi$ *(of a continuous invariant foliation for* $A(G) = \Pi$*) with fibres* $\{\gamma_{x'}\}_{x' \in B(\psi)}$ *that are pairwise disjoint graphs of continuous functions* $x = x_{x'}(y)$ *for every* $y \in pr_2(\Pi)$*. Moreover, the inclusion*

$$G(\gamma_{x'}) \subseteq \gamma_{\psi(x')} \text{ holds.} \quad (4)$$

The Analytic Criterion of the Integrability

Theorem 2. *Let* Π *be a compact trapezoid in the plane* $\mathbf{R^2}$, G *be a self-map of* Π, $A(G)$ *be a closed* G*-invariant subset of* Π *satisfying (3). Let* J *be a segment of the line* $\mathbf{R^1}$, ψ *be a self-map of* J, $B(\psi)$ *be a closed* ψ*-invariant subset of* J.
Then $G_{|A(G)}$ *is the geometrically integrable with the quotient* $\psi_{|B(\psi)}$ *by means of a continuous surjection* $H : A(G) \to B(\psi)$ *such that for every* $y \in pr_2(\Pi)$ *the map* H *is injection on* x, *if and only if there is a homeomorphism* $\widetilde{H}$ *that maps the set* $A(G)$ *on the set* $B(\psi) \times pr_2(\Pi)$ *and reduces the restriction* $G_{|A(G)}$ *to the skew product* $F_{|B(\psi)\times pr_2(\Pi)}$ *satisfying*

$$F_{|B(\psi)\times pr_2(\Pi)}(u, v) = (\psi_{|B(\psi)}(u), g_{x'}(v)), \;\; g_{x'}(v) = g(x', v), \quad (5)$$

where $x' = pr_1 \circ \widetilde{H}^{-1}(u, v)$, $pr_1 : \mathbf{R^2} \to Ox$ *is the first natural projection,* $\widetilde{H}^{-1} : B(\psi) \times J' \to A(G)$ *is the inverse homeomorphism for* $\widetilde{H}$, $J' = pr_2(\Pi)$.

Skew Products and Iterations of Integrable Maps

Consider the one-parameter family of skew products of interval maps $F_t : J_t \times J' \to J_t \times J'$, where $t \in J'$, $J_t = \{x : (x,\, t) \in \Pi\}$

$$F_t(x,\, y) = (g_t^1(x)),\, g_x^2(y).$$

Then integrable maps $G : \Pi \to \Pi$ $(G(x,\, y) = (g_y^1(x),\, g_x^2(y)))$ and skew products F_t coincide on a horizontal fibre $J_t \times \{t\}$:

$$G(x,\, t) = F_t(x,\, t)$$

Therefore, for every point $(x^0,\, y^0) \in J^2$ and every $n \geqslant 1$ we have:

$$G^n(x^0,\, y^0) = F_{y^{n-1}} \circ \ldots \circ F_{y^0}(x^0,\, y^0).$$

It means that an autonomous discrete dynamical system generated by an integrable map can be considered as a nonautonomous discrete dynamical system generated by the family $\{F_t\}_{t \in J'}$.
There is no a precise boundary between autonomous and nonautonomous discrete dynamical systems!!!

Extension of the Sharkovsky's Theorem on the Case of Integrable Maps

Theorem 3. *Let* Π *be a compact trapezoid in the plane* $\mathbf{R^2}$, G *be a continuous self-map of* Π, $A(G)$ *be a closed* G*-invariant subset of* Π *containing the set* $Per(G)$ *and satisfying (3). Let* J *be a closed interval of the line* $\mathbf{R^1}$, ψ *be a continuous self-map of* J, $B(\psi)$ *be a closed* ψ*-invariant subset of* J. *Let* $G_{|A(G)}$ *be geometrically integrable with the quotient* $\psi_{|B(\psi)}$ *by means of a continuous surjection* $H : A(G) \to B(\psi)$ *such that for every* $y \in J'$ *the map* H *be an injection on* x, *and* G *contain a periodic point with (least) period* $m > 1$.
Then G *contains also periodic points of every (least) period* n, *where* n *precedes* m *in the Sharkovsky's order*

$$1 \prec 2 \prec 2^2 \prec 2^3 \prec \ldots \prec \ldots \prec 2^2 \cdot 9 \prec 2^2 \cdot 7 \prec 2^2 \cdot 5 \prec 2^2 \cdot 3 \prec \ldots$$
$$\prec 2 \cdot 9 \prec 2 \cdot 7 \prec 2 \cdot 5 \prec 2 \cdot 3 \prec \ldots \prec 9 \prec 7 \prec 5 \prec 3.$$

The Important Corollary

Corollary. *Let* Π *be a compact trapezoid in the plane* $\mathbf{R^2}$*,* G *be a continuous self-map of* Π*. Let* J *be a closed interval of the line* $\mathbf{R^1}$*,* ψ *be a continuous self-map of* J*. Let, in addition,* G *be geometrically integrable with the quotient* ψ *by means of a continuous surjection* $H : \Pi \to J$ *such that for every* $y \in J'$ *the map* H *be an injection on* x*, and* G *contain a periodic point with (least) period* $m > 1$*.*
Then G *contains also periodic points of every (least) period* n*, where* n *precedes* m *in the Sharkovsky's order*

$$1 \prec 2 \prec 2^2 \prec 2^3 \prec \ldots \prec \ldots \prec 2^2 \cdot 9 \prec 2^2 \cdot 7 \prec 2^2 \cdot 5 \prec 2^2 \cdot 3 \prec \ldots$$
$$\prec 2 \cdot 9 \prec 2 \cdot 7 \prec 2 \cdot 5 \prec 2 \cdot 3 \prec \ldots \prec 9 \prec 7 \prec 5 \prec 3.$$

Lorenz Maps Derived from Symmetric Unimodal Maps

Symmetry of a unimodal map $f : [a,\, b] \to [a,\, b]$ means that for every $x \in [a,\, b]$ the equality holds

$$f(x) = f(a + b - x).$$

Let a symmetric unimodal map $f : [a,\, b] \to [a,\, b]$ with the unique critical point $c = (a + b)/2$ be so that $f(c) = b$.
Define an increasing symmetric Lorenz map $\varphi : [a,\, b] \to [a,\, b]$ setting

$$\varphi(x) = \begin{cases} f(x), & \text{if} \quad x \in [a,\, c]; \\ a + b - f(x), & \text{if} \quad x \in (c,\, b]. \end{cases}$$

Define a decreasing symmetric Lorenz map $\psi : [a,\, b] \to [a,\, b]$ setting

$$\psi(x) = \begin{cases} a + b - f(x), & \text{if} \quad x \in [a,\, c]; \\ f(x), & \text{if} \quad x \in (c,\, b]. \end{cases}$$

Periodic Points Periods of Lorenz Maps Derived from Symmetric Unimodal Maps

[10] A. Anušić, H. Bruin, and J. Činč, "Topological properties of Lorenz maps derived from unimodal maps," J. Diff. Eq. and Applic. **26** (8), Special issue on the occasion of the 82nd birthday of Oleksandr M. Sharkovsky, 1174–1191 (2020).

Theorem [10]. *Symmetric increasing Lorenz maps φ satisfy Sharkovsky's Theorem, except for the fixed points.*

Decreasing symmetric Lorenz maps ψ satisfy Sharkovsky's Theorem, possibly except for periods 2^r, $r \geqslant 1$.

Periodic Points of Integrable Maps with Lorenz Quotients, I (conditions of Theorem 4)

Theorem 4. *Let Π be a compact curvilinear trapezoid in the plane $\mathbf{R^2}$, G be a self-map of Π with a continuous second coordinate function.*

Let J be a segment of the line $\mathbf{R^1}$, ψ be a Lorenz self-map of J derived from a symmetric unimodal map.

Let, in addition, G be geometrically integrable with the quotient ψ by means of a continuous surjection $H : \Pi \to J$ such that for every $y \in J'$ the map H be an injection on x, and G contain a periodic point with (least) period $m > 1$.

Periodic Points of Integrable Maps with Lorenz Quotients, II (the conclusion of Theorem 4)

Then G contains also periodic points of every (least) period n, where n precedes m in the Sharkovsky's order, except for $n = 1$, if ψ is a symmetric increasing Lorenz map;
G contains periodic points of every (least) period n, where n precedes m in the Sharkovsky's order, possibly except for periods 2^r, $r \geqslant 1$, if ψ is a symmetric decreasing Lorenz map.

If ψ is a symmetric decreasing Lorenz map, and ψ has no periodic points with periods 2^r, $r \geqslant 1$, then G has periodic points with periods 2^r, $r \geqslant 1$, if and only if there is G-invariant curvilinear fibre $\gamma_{x'}$ for $x' \in Fix(\psi)$, where $Fix(\cdot)$ is the set of fixed points of a map, that contains periodic points of the map $g^2_{x_{x'}} : J' \to J'$ with periods 2^r, $r \geqslant 1$. Here $x_{x'}$ is a function with the graph $\gamma_{x'}$.

[11] L.E. and E. Makhrova, "One-dimensional dynamical systems", Russian Math. Survey, **76**:5 (2021), 81–146.

The Sufficient Condition of the Topological Entropy Positivity

Proposition 1. *Let Π be a compact curvilinear trapezoid in the plane $\mathbf{R}^2$, $G : \Pi \to \Pi$ be an integrable map satisfying conditions of Theorem 3 or Theorem 4. Let, in addition, G have a periodic orbit with a (least) period $m \notin \{2^r\}_{r \geqslant 0}$. Then the topological entropy $h(G)$ of the map G is positive (i.e. G admits the entropy chaos).*

There is the example of a continuous geometrically integrable map in the plane with a positive topological entropy and the set of (least) periods of periodic points of the type

$$\tau(G) = \{2^r\}_{r \geqslant 0}.$$

Is there an example of a C^1-smooth geometrically integrable map with the above properties?