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Periodic Motion of a Swinging Atwood's Machine

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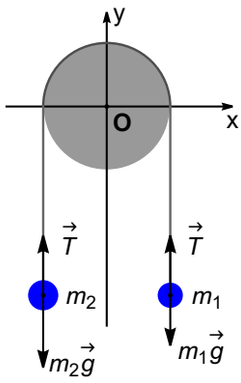
Introduction

An Atwood's machine is a well-known device that is usually used in the course in physics to demonstrate the uniformly accelerated motion of a system. It consists of two bodies, having masses m_1, m_2 ($m_2 \geq m_1$), attached to opposite ends of a massless inextensible thread wound round a massless, frictionless pulley.



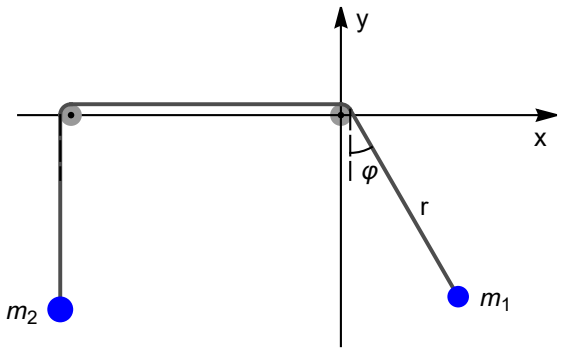
It is assumed that each body can move only along vertical axis *Oy*, and the thread doesn't slip on the pulley. Such Atwood's machine is a simple mechanical system with one degree of freedom and, using Newton's second law, one can easily obtain its equation of motion and show that acceleration of each body is given by

$$a = \frac{m_2 - m_1}{m_2 + m_1}\, g,$$

where g is a gravitational acceleration.

Physical Problem

The Atwood machine under consideration is shown below. The pulley of finite radius and mass is replaced by two small massless pulleys located on some distance from each other. This separation of the pulleys is necessary to avoid collisions of the bodies during oscillation. The system has two degrees of freedom and its geometrical configuration can be described in terms of two variables, namely, a length r of the thread between the body m_1 and the point where the thread departs from the pulley, and the angle φ determining deviation of the thread from a vertical.



One can readily see that Cartesian coordinates x_1, x_2, y_1, y_2 of the bodies can be determined as

```
In[•]:= x1[t_] = r[t] Sin[φ[t]];
        y1[t_] = -r[t] Cos[φ[t]];
        x2[t_] = -b;
        y2[t_] = -(L - r[t]);
```

where b is a distance between the pulleys.

Equations of Motion

Doing standard symbolic computation,the Lagrangian function of the system can be written in the form

```
In[•]:= Lag = (-1 - ε + Cos[φ[t]]) r[t] + 1/2 (2 + ε) r′[t]^2 + 1/2 r[t]^2 φ′[t]^2;
```

where parameter $\varepsilon = (m_2 - m_1)/m_1$ determines a difference of masses.

Then equations of motion are given by

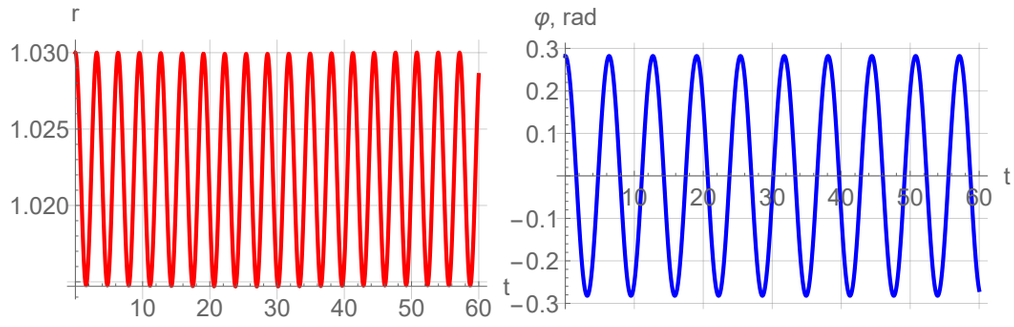
```
In[•]:= eq = {D[D[Lag, r'[t]], t] == D[Lag, r[t]],
             D[D[Lag, φ'[t]], t] == D[Lag, φ[t]]};
        eq1 =
         {r''[t], φ''[t]} ==
           ({r''[t], φ''[t]} /. Solve[eq, {r''[t], φ''[t]}][[1]] //
             Collect[#, {r′[t], φ′[t]}, Simplify] &) // Thread
```

Out[•]= $\left\{r''[t] == \frac{-1 - \varepsilon + \text{Cos}[\varphi[t]]}{2 + \varepsilon} + \frac{r[t]\,\varphi'[t]^2}{2 + \varepsilon},\ \varphi''[t] == -\frac{\text{Sin}[\varphi[t]]}{r[t]} - \frac{2\,r'[t]\,\varphi'[t]}{r[t]}\right\}$

Choosing some initial conditions, we can solve equations of motion numerically and show that the system can demonstrate periodic motion.

In[•]:=
```
rul0 = {ε → 0.02};
initial0 = {φ[0] == 2 √ε - 0.00077, φ'[0] == 0, r[0] == 1.03, r'[0] == 0};
sol0 = NDSolve[Join[eq1, initial0] /. rul0, {φ, r}, {t, 0, 200}][[1]];
Row[{Plot[{r[t] /.sol0} // Evaluate, {t, 0, 60}, AxesLabel → {"t", "r"},
    PlotStyle → {Thick, Red}, BaseStyle → {FontFamily → "Arial", FontSize → 12},
    ImageSize → 250, GridLines -> Automatic],
  Plot[{φ[t] /.sol0}, {t, 0, 60}, AxesLabel → {"t", "φ, rad"},
    PlotStyle → {{Thick, Blue}},
    BaseStyle → {FontFamily → "Arial", FontSize → 12}, ImageSize → 250,
    GridLines -> Automatic, PlotRange → All]}]
```

Out[•]=



Analytical Solution

First, we replace trigonometric functions by their expansions in power series.

In[•]:=
```
eq2 =
  eq1 /. {Cos[x_] → 1 - x^2 / 2 + x^4 / 24 - x^6 / 720,
     Sin[x_] → x - x^3 / 6 + x^5 / 120 - x^7 / 5040} // Simplify;
{r''[t], φ''[t]} ==
  ({r''[t], φ''[t]} /. Solve[eq2, {r''[t], φ''[t]}][[1]] // Expand) //
 Thread
```

Out[•]=
$$\left\{r''[t] == -\frac{\varepsilon}{2+\varepsilon} - \frac{\varphi[t]^2}{2\,(2+\varepsilon)} + \frac{\varphi[t]^4}{24\,(2+\varepsilon)} - \frac{\varphi[t]^6}{720\,(2+\varepsilon)} + \frac{r[t]\,\varphi'[t]^2}{2+\varepsilon},\ \varphi''[t] == -\frac{\varphi[t]}{r[t]} + \frac{\varphi[t]^3}{6\,r[t]} - \frac{\varphi[t]^5}{120\,r[t]} + \frac{\varphi[t]^7}{5040\,r[t]} - \frac{2\,r'[t]\,\varphi'[t]}{r[t]}\right\}$$

We look for a state of dynamic equilibrium

In[•]:=
```
rul1 = {r → (1 + ε r[#] &), φ → (√ε φ[#] &)};
```

On substituting ***rul1*** into ***eq2***, we rewrite the equations of motion in the form

In[•]:=
```
eq3a = (eq2[[1, 1]] - eq2[[1, 2]]) (2 + ε) / 1440 / ε /. rul1 // Expand // Collect[#, ε] &
```

Out[•]=
$$\frac{1}{2} + \frac{\varphi[t]^2}{4} + \frac{\varepsilon^2\,\varphi[t]^6}{1440} - \frac{1}{2}\,\varphi'[t]^2 + \varepsilon\left(-\frac{1}{48}\,\varphi[t]^4 - \frac{1}{2}\,r[t]\,\varphi'[t]^2 + \frac{r''[t]}{2}\right) + r''[t]$$

In[•]:=
```
eq3b = (eq2[[2, 1]] - eq2[[2, 2]]) / √ε /. rul1 // Expand
```

Out[•]=
$$\frac{\varphi[t]}{1+\varepsilon\,r[t]} - \frac{\varepsilon\,\varphi[t]^3}{6\,(1+\varepsilon\,r[t])} + \frac{\varepsilon^2\,\varphi[t]^5}{120\,(1+\varepsilon\,r[t])} - \frac{\varepsilon^3\,\varphi[t]^7}{5040\,(1+\varepsilon\,r[t])} + \frac{2\,\varepsilon\,r'[t]\,\varphi'[t]}{1+\varepsilon\,r[t]} + \varphi''[t]$$

We look for a solution of the form:

In[•]:=
```
rul2 = {r → (r0[#] + ε r1[#] + ε^2 r2[#] + ε^3 r3[#] &),
        φ → (φ0[#] + ε φ1[#] + ε^2 φ2[#] + ε^3 φ3[#] &)};
```

Now we can substitute this solution into the equations of motion and expand them into power series in terms of a small parameter ε.

```
In[•]:= eq4a = eq3a /. rul2 // Series[#, {ε, 0, 3}] & // Normal
```

Out[•]=
$$\frac{1}{4}\left(2+\varphi0[t]^2-2\,\varphi0'[t]^2+4\,r0''[t]\right)+\frac{1}{48}\,\varepsilon\left(-\varphi0[t]^4+24\,\varphi0[t]\times\varphi1[t]-24\,r0[t]\,\varphi0'[t]^2-48\,\varphi0'[t]\,\varphi1'[t]+24\,r0''[t]+48\,r1''[t]\right)+\frac{1}{1440}\,\varepsilon^2\left(\varphi0[t]^6-120\,\varphi0[t]^3\,\varphi1[t]+360\,\varphi1[t]^2+720\,\varphi0[t]\times\varphi2[t]-720\,r1[t]\,\varphi0'[t]^2-1440\,r0[t]\,\varphi0'[t]\,\varphi1'[t]-720\,\varphi1'[t]^2-1440\,\varphi0'[t]\,\varphi2'[t]+720\,r1''[t]+1440\,r2''[t]\right)+\frac{1}{240}\,\varepsilon^3\left(\varphi0[t]^5\,\varphi1[t]-30\,\varphi0[t]^2\,\varphi1[t]^2-20\,\varphi0[t]^3\,\varphi2[t]+120\,\varphi1[t]\times\varphi2[t]+120\,\varphi0[t]\times\varphi3[t]-120\,r2[t]\,\varphi0'[t]^2-240\,r1[t]\,\varphi0'[t]\,\varphi1'[t]-120\,r0[t]\,\varphi1'[t]^2-240\,r0[t]\,\varphi0'[t]\,\varphi2'[t]-240\,\varphi1'[t]\,\varphi2'[t]-240\,\varphi0'[t]\,\varphi3'[t]+120\,r2''[t]+240\,r3''[t]\right)$$

```
In[•]:= eq4b = φ''[t] + φ[t] - eq3b /. rul2 // Series[#, {ε, 0, 3}] & // Normal
```

Out[•]=
$$\varepsilon\left(r0[t]\times\varphi0[t]+\frac{\varphi0[t]^3}{6}-2\,r0'[t]\,\varphi0'[t]\right)+\varepsilon^2\left(-\left(\left(r0[t]^2-r1[t]\right)\varphi0[t]\right)-\frac{\varphi0[t]^5}{120}+r0[t]\times\varphi1[t]+\frac{1}{6}\left(-r0[t]\,\varphi0[t]^3+3\,\varphi0[t]^2\,\varphi1[t]\right)-2\left(\left(-r0[t]\,r0'[t]+r1'[t]\right)\varphi0'[t]+r0'[t]\,\varphi1'[t]\right)\right)+\varepsilon^3\left(r0[t]^3\,\varphi0[t]-2\,r0[t]\times r1[t]\times\varphi0[t]+r2[t]\times\varphi0[t]+\frac{\varphi0[t]^7}{5040}-r0[t]^2\,\varphi1[t]+r1[t]\times\varphi1[t]+\frac{1}{120}\left(r0[t]\,\varphi0[t]^5-5\,\varphi0[t]^4\,\varphi1[t]\right)+r0[t]\times\varphi2[t]+\frac{1}{6}\left(\left(r0[t]^2-r1[t]\right)\varphi0[t]^3-3\,r0[t]\,\varphi0[t]^2\,\varphi1[t]+3\,\varphi0[t]\left(\varphi1[t]^2+\varphi0[t]\times\varphi2[t]\right)\right)-2\left(\left(\left(r0[t]^2-r1[t]\right)r0'[t]-r0[t]\,r1'[t]+r2'[t]\right)\varphi0'[t]+\left(-r0[t]\,r0'[t]+r1'[t]\right)\varphi1'[t]+r0'[t]\,\varphi2'[t]\right)\right)$$

Equating coefficients of ε^k, $k = 0, 1, 2, \ldots$ in each equation, we obtain a system of differential equations which can be solved in succession.

We assume that at the initial instant of time the pendulum is released with zero initial velocity. Then in zero approximation we obtain a solution

```
In[•]:= eqφ[φ_] := φ''[t] + φ[t]
        solφ0 = DSolve[{eqφ[φ0] == 0, φ0'[0] == 0}, φ0[t], t] /. {C[1] → A0} // First
```

Out[•]= $\{\varphi0[t]\to A0\,\mathrm{Cos}[t]\}$

Next step - we substitute $\varphi0$ into the coefficient of ε in ***eq4a*** and obtain

```
In[•]:= eq5a = Coefficient[eq4a, ε, 0] /. solφ0 /. D[solφ0, t] // Expand // TrigReduce
```

Out[•]=
$$\frac{1}{8}\left(4-A0^2+3\,A0^2\,\mathrm{Cos}[2\,t]+8\,r0''[t]\right)$$

To avoid of increase of the distance *r0* we have to choose parameter c_1 in such a way that constant terms in ***eq5a*** vanish.

```
In[•]:= solA0 = A0 → 2;
In[•]:= sol5a = DSolve[{(eq5a /. solA0) == 0, r0'[0] == 0}, r0[t], t] /. C[1] → C1 // First // Expand
```

Out[•]=
$$\left\{r0[t]\to C1+\frac{3}{8}\,\mathrm{Cos}[2\,t]\right\}$$

Next step - we substitute *r0* into the coefficient of ε in ***eq4b*** and obtain

```
In[•]:= eq5b = Coefficient[eq4b, ε, 1] /. sol5a /. D[sol5a, t] /. solφ0 /. D[solφ0, t] /. solA0 //
          Expand // TrigReduce
```

Out[•]= $\frac{1}{24}\left(-3\,\mathrm{Cos}[t] + 48\,C1\,\mathrm{Cos}[t] + 53\,\mathrm{Cos}[3\,t]\right)$

To avoid resonance we have to choose *C1* in such a way that coefficient of ***cost*** is equal to zero.

```
In[•]:= solC1 = Solve[(Coefficient[eq5b, Cos[t]]) == 0, C1] // First
```

Out[•]= $\left\{C1 \to \frac{1}{16}\right\}$

Thus, in the first approximation we obtain

```
In[•]:= solr0 = sol5a /. solC1 // Expand
```

Out[•]= $\left\{r0[t] \to \frac{1}{16} + \frac{3}{8}\,\mathrm{Cos}[2\,t]\right\}$

```
In[•]:= sol5b = DSolve[{eqφ[φ1] == (eq5b /. solC1), φ1'[0] == 0}, φ1[t], t] /. C[1] → C2 // First //
          TrigReduce
```

Out[•]= $\left\{\varphi1[t] \to \frac{1}{192}\left(-106\,\mathrm{Cos}[t] + 192\,C2\,\mathrm{Cos}[t] - 53\,\mathrm{Cos}[3\,t]\right)\right\}$

We repeat this procedure step by step.
Finally, we obtain a solution accurate to the third order in ε.

In[•]:= $$rr[t_] = 1 + \varepsilon\left(\frac{1}{16} + \frac{3}{8}\,\mathrm{Cos}[2\,t] + \varepsilon\left(-\frac{261}{2048} - \frac{69}{512}\,\mathrm{Cos}[2\,t] - \frac{105\,\mathrm{Cos}[4\,t]}{2048}\right) + \varepsilon^2\left(\frac{4275}{131\,072} + \frac{4083\,\mathrm{Cos}[2\,t]}{65\,536} + \frac{5067\,\mathrm{Cos}[4\,t]}{131\,072} + \frac{755\,\mathrm{Cos}[6\,t]}{65\,536}\right)\right);$$

In[•]:= $$\varphi\varphi[t_] = \sqrt{\varepsilon}\left(2\,\mathrm{Cos}[t] - \frac{53}{192}\,\varepsilon\,\mathrm{Cos}[3\,t] + \varepsilon^2\left(\frac{2959\,\mathrm{Cos}[t]}{16\,384} + \frac{1699\,\mathrm{Cos}[3\,t]}{16\,384} + \frac{5813\,\mathrm{Cos}[5\,t]}{81\,920}\right) + \varepsilon^3\left(-\frac{1\,129\,411\,\mathrm{Cos}[t]}{18\,874\,368} - \frac{251\,267\,\mathrm{Cos}[3\,t]}{2\,097\,152} - \frac{86\,615\,\mathrm{Cos}[5\,t]}{1\,572\,864} - \frac{961\,939\,\mathrm{Cos}[7\,t]}{44\,040\,192}\right)\right);$$

Note that initial values of the variables are

```
In[•]:= {rr[0], φφ[0]}
```

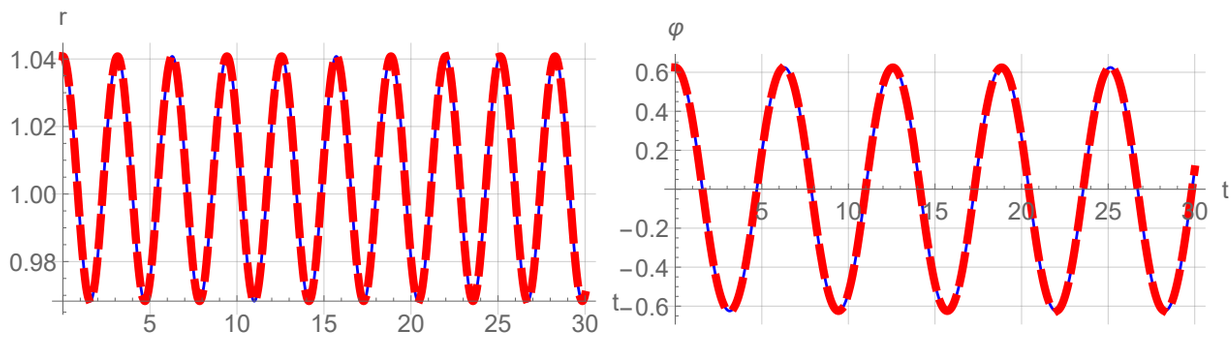
Out[•]= $\left\{1 + \varepsilon\left(\frac{7}{16} - \frac{321\,\varepsilon}{1024} + \frac{9509\,\varepsilon^2}{65\,536}\right),\ \sqrt{\varepsilon}\left(2 - \frac{53\,\varepsilon}{192} + \frac{29\,103\,\varepsilon^2}{81\,920} - \frac{33\,897\,175\,\varepsilon^3}{132\,120\,576}\right)\right\}$

Now we can compare analytical solutions found above and the corresponding numerical solutions.

```
In[•]:= rul6 = {ε → 0.1, g → 9.8, R0 → 1};
        initial6 = {φ[0] == φφ[0], φ'[0] == 0,
                    r[0] == rr[0], r'[0] == 0};
        sol16 = NDSolve[Join[eq2, initial6] /. rul6 , {φ, r}, {t, 0, 200}][[1]];
```

In[∘]:=
```
Row[{Plot[{r[t] /.sol16 , rr[t] /. rul6 } // Evaluate, {t, 0, 30},
    AxesLabel → {"t", "r"},
    PlotStyle → {{Thickness[0.005], Blue},
      {Thickness[0.015], Dashing[{0.05, 0.03}], Red}},
    BaseStyle → {FontFamily → "Arial", FontSize → 12}, ImageSize → 300,
    GridLines -> Automatic, AspectRatio → 1 / 2],
  Plot[{φ[t] /.sol16, φφ[t] /. rul6} // Evaluate, {t, 0, 30},
    AxesLabel → {"t", "φ"},
    PlotStyle → {{Thickness[0.005], Blue},
      {Thickness[0.015], Dashing[{0.05, 0.03}], Red}},
    BaseStyle → {FontFamily → "Arial", FontSize → 12}, ImageSize → 300,
    GridLines -> Automatic, PlotRange → All, AspectRatio → 1 / 2]}]
```

Out[∘]=



Stability Analysis

Doing quite standard symbolic calculation, we derive the Hamiltonian of the system

In[∘]:=
$$H0 = \frac{pr[t]^2}{2\,(2+\varepsilon)} + \frac{p\varphi[t]^2}{2\,r[t]^2} - r[t]\left(-\varepsilon - \frac{\varphi[t]^2}{2} + \frac{\varphi[t]^4}{24} - \frac{\varphi[t]^6}{720}\right);$$

Let us add small perturbations to the periodic solution

In[∘]:=
```
rulδ = {pr[t] → (pr0[t] + δ p1[t]), pφ[t] → (pφ0[t] + δ p2[t]),
    r[t] → (rp[t] + δ q1[t]), φ[t] → (φp[t] + δ q2[t])};
```

Making the corresponding canonical transformation, we obtain the Hamiltonian in the form of power series with respect to the new cannical variables

In[∘]:=
```
H0exp =
  (H0 /. rulδ) + δ q1[t] × pr0'[t] + δ pφ0'[t] × q2[t] - δ rp'[t] × p1[t] -
      δ φp'[t] × p2[t] // Series[#, {δ, 0, 2}] & // Normal //
   Collect[#, {δ, q1[t], q2[t], p1[t], p2[t]}] &
```

Out[∘]=
$$\frac{pr0[t]^2}{2\,(2+\varepsilon)} + \frac{p\varphi0[t]^2}{2\,rp[t]^2} + \frac{1}{720}\,rp[t]\left(720\,\varepsilon + 360\,\varphi p[t]^2 - 30\,\varphi p[t]^4 + \varphi p[t]^6\right) +$$
$$\delta^2\left(\frac{p1[t]^2}{2\,(2+\varepsilon)} + \frac{3\,p\varphi0[t]^2\,q1[t]^2}{2\,rp[t]^4} + \frac{p2[t]^2}{2\,rp[t]^2} + q2[t]^2\left(\frac{rp[t]}{2} - \frac{1}{4}\,rp[t]\,\varphi p[t]^2 + \frac{1}{48}\,rp[t]\,\varphi p[t]^4\right) +\right.$$
$$\left. q1[t]\left(-\frac{2\,p2[t]\times p\varphi0[t]}{rp[t]^3} + q2[t]\left(\varphi p[t] - \frac{\varphi p[t]^3}{6} + \frac{\varphi p[t]^5}{120}\right)\right)\right) +$$
$$\delta\left(q1[t]\left(-\frac{p\varphi0[t]^2}{rp[t]^3} + \frac{1}{720}\left(720\,\varepsilon + 360\,\varphi p[t]^2 - 30\,\varphi p[t]^4 + \varphi p[t]^6\right) + pr0'[t]\right) +\right.$$
$$\left. q2[t]\left(rp[t]\times\varphi p[t] - \frac{1}{6}\,rp[t]\,\varphi p[t]^3 + \frac{1}{120}\,rp[t]\,\varphi p[t]^5 + p\varphi0'[t]\right) +\right.$$

$$p1[t]\left(\frac{pr0[t]}{2+\varepsilon}-rp'[t]\right)+p2[t]\left(\frac{p\varphi0[t]}{rp[t]^2}-\varphi p'[t]\right)\Bigg)$$

Zero order terms of the Hamiltonian are functions of time which do not appear in the equations of motion and may be omitted. . The first order terms are equal to zero due to the unperturbed equations of motion. Finally, the second order term in the Hamiltonian expansion is

```
In[•]:= H20 = Coefficient[H0exp, δ, 2] // Collect[#, {p1[t], p2[t], q1[t], q2[t]}] &
```

Out[•]=

$$\frac{p1[t]^2}{2\,(2+\varepsilon)}+\frac{3\,p\varphi0[t]^2\,q1[t]^2}{2\,rp[t]^4}-\frac{2\,p2[t]\times p\varphi0[t]\times q1[t]}{rp[t]^3}+\frac{p2[t]^2}{2\,rp[t]^2}+q2[t]^2\left(\frac{rp[t]}{2}-\frac{1}{4}\,rp[t]\,\varphi p[t]^2+\frac{1}{48}\,rp[t]\,\varphi p[t]^4\right)+q1[t]\times q2[t]\left(\varphi p[t]-\frac{\varphi p[t]^3}{6}+\frac{\varphi p[t]^5}{120}\right)$$

Remind that periodic solution is represented in the form of power series in ε.

```
In[•]:= rulOld = {rp → (1 + ε (r0[#] + ε r1[#] + ε^2 r2[#]) &),
    φp → (√ε (φ0[#] + ε φ1[#] + ε^2 φ2[#]) &)};
```

The corresponding conjugate momenta are

```
In[•]:= rulOldP = {pr0[t] → ((2 + ε) rp'[t]), pφ0[t] → rp[t]^2 × φp'[t]};
```

Therefore, the Hamiltonian may be written in the form

```
In[•]:= H2exp = H20 /. rulOldP /. rulOld // Series[#, {ε, 0, 3}] & // Normal //
   Collect[#, {p1[t], p2[t], q1[t], q2[t], ε}, Simplify] &
```

The quadratic part ***H2exp*** of the Hamiltonian determines the linearized equations of the perturbed motion which is convenient to write in the matrix form

$$\frac{dx}{dt}=J\cdot H\,(t,\,\varepsilon)\cdot x,$$

where $J=\begin{pmatrix}0 & E_2\\ -E_2 & 0\end{pmatrix}$, E_2 is the second order identity matrix, a 4-dimentional vector $\boldsymbol{x}$ and the fourth-order matrix-function $H(t,\,\varepsilon)=[H_{ij}]$ are given by

$$x^T=(q1\,(t),\,q2\,(t),\,p1\,(t),\,p2\,(t));\;\; H_{ij}=\frac{\partial^2 H_{2\,exp}}{\partial x_i\,\partial x_j};\;\; i,\,j=1,\,2,\,3,\,4.$$

It is clear that matrix H(t,ε) is periodic function of time and so the perturbed motion of the system is described by the linear system of four differential equations with periodic coefficients.

Behavior of solutions of the system above is determined by its characteristic multipliers which are the eigenvaues of the monodromy matrix $X(2\,\pi,\,\varepsilon)$, where $X(t,\,\varepsilon)$ is a fundamental matrix for the system satisfying the initial condition $X(0,\,\varepsilon)=E_4$. As periodic solution we consider is represented by power series in ε, the matrix $H(t,\,\varepsilon)$ can also be represented in the form of power series

$$H\,(t,\,\varepsilon)=\sum_{k=0}^{\infty}H_k\,(t)\,\varepsilon^{k/2}.$$

The matrix-functions $H_k(t)$ are given by

```
In[•]:= x[t_] = {q1[t], q2[t], p1[t], p2[t]};
matH2 = Table[D[H2exp, x[t][[j]], x[t][[k]] ], {j, 4}, {k, 4}] // Simplify;
```

$$\text{In[•]:= } \text{rulR}\varphi = \left\{r0 \to \left(\left(\frac{1}{16} - \frac{3}{8}\,\text{Cos}[2\,\#]\right)\&\right),\ r1 \to \left(\frac{-261 + 276\,\text{Cos}[2\,\#] - 105\,\text{Cos}[4\,\#]}{2048}\,\&\right),\right.$$
$$r2 \to \left(\frac{4275 - 8166\,\text{Cos}[2\,\#] + 5067\,\text{Cos}[4\,\#] - 1510\,\text{Cos}[6\,\#]}{131\,072}\,\&\right),$$
$$\varphi 0 \to (2\,\text{Sin}[\#]\,\&),\ \varphi 1 \to \left(\frac{53}{192}\,\text{Sin}[3\,\#]\,\&\right),$$
$$\left.\varphi 2 \to \left(\frac{2959\,\text{Sin}[\#] + 1699\,\text{Sin}[3\,\#]}{16\,384} + \frac{5813}{81\,920}\,\text{Sin}[5\,\#]\,\&\right)\right\};$$

```
In[•]:= matH2t = (matH2 /. rulRφ // Collect[#, ε, Simplify] & )
```

$$\text{Out[•]= } \left\{\left\{12\,\varepsilon\,\text{Cos}[t]^2 + \frac{159}{16}\,\varepsilon^2\,\text{Cos}[t]\,\text{Cos}[3\,t] + \frac{3\,\varepsilon^3\,\text{Cos}[t]^2\,(6051 - 6334\,\text{Cos}[2\,t] + 8622\,\text{Cos}[4\,t])}{2048},\right.\right.$$
$$2\,\sqrt{\varepsilon}\,\text{Sin}[t] + \varepsilon^{3/2}\left(-\text{Sin}[t] + \frac{39}{64}\,\text{Sin}[3\,t]\right) + \frac{\varepsilon^{5/2}\,(7951\,\text{Sin}[t] - 4189\,\text{Sin}[3\,t] + 3697\,\text{Sin}[5\,t])}{16\,384},\ 0,$$
$$\left.-4\,\sqrt{\varepsilon}\,\text{Cos}[t] + \varepsilon^{3/2}\left(-\frac{\text{Cos}[t]}{2} - \frac{77}{32}\,\text{Cos}[3\,t]\right) + \frac{\varepsilon^{5/2}\,(-9135\,\text{Cos}[t] - 3265\,\text{Cos}[3\,t] - 10\,349\,\text{Cos}[5\,t])}{8192}\right\},$$
$$\left\{2\,\sqrt{\varepsilon}\,\text{Sin}[t] + \varepsilon^{3/2}\left(-\text{Sin}[t] + \frac{39}{64}\,\text{Sin}[3\,t]\right) + \frac{\varepsilon^{5/2}\,(7951\,\text{Sin}[t] - 4189\,\text{Sin}[3\,t] + 3697\,\text{Sin}[5\,t])}{16\,384},\right.$$
$$1 + \frac{5}{16}\,\varepsilon\,(-3 + 2\,\text{Cos}[2\,t]) + \frac{\varepsilon^2\,(-261 - 76\,\text{Cos}[2\,t] + 247\,\text{Cos}[4\,t])}{2048} +$$
$$\left.\frac{\varepsilon^3\,(658\,615 - 1\,972\,590\,\text{Cos}[2\,t] + 1\,113\,279\,\text{Cos}[4\,t] + 185\,666\,\text{Cos}[6\,t])}{5\,898\,240},\ 0,\ 0\right\},$$
$$\left\{0,\ 0,\ \frac{1}{2} - \frac{\varepsilon}{4} + \frac{\varepsilon^2}{8} - \frac{\varepsilon^3}{16},\ 0\right\},\ \left\{-4\,\sqrt{\varepsilon}\,\text{Cos}[t] + \varepsilon^{3/2}\left(-\frac{\text{Cos}[t]}{2} - \frac{77}{32}\,\text{Cos}[3\,t]\right) +\right.$$
$$\frac{\varepsilon^{5/2}\,(-9135\,\text{Cos}[t] - 3265\,\text{Cos}[3\,t] - 10\,349\,\text{Cos}[5\,t])}{8192},\ 0,$$
$$0,\ 1 + \frac{1}{8}\,\varepsilon\,(-1 + 6\,\text{Cos}[2\,t]) + \frac{3\,\varepsilon^2\,(163 - 140\,\text{Cos}[2\,t] + 107\,\text{Cos}[4\,t])}{1024} +$$
$$\left.\left.\frac{\varepsilon^3\,(-20\,863 + 45\,570\,\text{Cos}[2\,t] - 19\,719\,\text{Cos}[4\,t] + 8746\,\text{Cos}[6\,t])}{65\,536}\right\}\right\}$$

The fundamental matrix $X(t, \varepsilon)$ can be sought in the form of power series

$$X\,(t,\ \varepsilon) = \sum_{k=0}^{\infty} X_k\,(t)\,\varepsilon^{k/2},$$

where $X_k(t)$ are continuous matrix-functions.
On substituting the matrices $X(t, \varepsilon)$ and $H(t, \varepsilon)$ into the linearized system above and collecting coefficients of equal power of ε, we obtain the sequence of differential equations of the form

$$\frac{dX_0}{dt} = J \cdot H_0 \cdot X_0\,(t),$$

$$\frac{dX_k}{dt} - J \cdot H_0 \cdot X_k\,(t) = \sum_{j=1}^{k} J \cdot H_j\,(t) \cdot X_{k-j}\,(t),\quad (k >= 1).$$

The functions $X_k(t)$ must satisfy the following initial conditions:

$$X_0\,(0) = E_4,\quad X_k\,(0) = 0,\quad (k >= 1).$$

Formally, we can represent a solution to the first equation in the form

$$X_0\,(t) = \exp\,(J \cdot H_0\,t).$$

So it is convenient to represent solutions $X_k(t)$ in the form

$$X_k(t) = \exp(J \cdot H_0 t) \cdot Y_k(t).$$

Then matrix-functions $Y_k(t)$ are determined by the equations

$$\frac{dY_k}{dt} = \sum_{j=1}^{k} \exp(-J \cdot H_0 t) \cdot J \cdot H_j(t) \exp(J \cdot H_0 t) \cdot Y_{k-j}(t), \quad (k >= 1).$$

The functions $Y_k(t)$ must satisfy the initial conditions

$$Y_k(0) = 0, \quad (k >= 1).$$

The equations are solved in succession. First, we compute the fundamental matrix $X_0(t)$ in case of $\varepsilon = 0$.

```
In[•]:= eq35a = q1'[t] - D[H2exp /. ε → 0, p1[t]]
```

Out[•]= $-\frac{p1[t]}{2} + q1'[t]$

```
In[•]:= eq35b = q2'[t] - D[H2exp /. ε → 0, p2[t]]
```

Out[•]= $-p2[t] + q2'[t]$

```
In[•]:= eq35c = p1'[t] + D[H2exp /. ε → 0, q1[t]]
```

Out[•]= $p1'[t]$

```
In[•]:= eq35d = p2'[t] + D[H2exp /. ε → 0, q2[t]]
```

Out[•]= $q2[t] + p2'[t]$

Equations in zero approximation

```
In[•]:= eqP0 = {eq35a == 0, eq35b == 0, eq35c == 0, eq35d == 0}
```

Out[•]= $\left\{-\frac{p1[t]}{2} + q1'[t] == 0, -p2[t] + q2'[t] == 0, p1'[t] == 0, q2[t] + p2'[t] == 0\right\}$

```
In[•]:= solP0a = DSolve[Join[eqP0, {q1[0] == 1, q2[0] == 0, p1[0] == 0, p2[0] == 0}], {q1, q2, p1, p2}, t] //
  First
```

Out[•]= {p1 → Function[{t}, 0], q1 → Function[{t}, 1], p2 → Function[{t}, 0], q2 → Function[{t}, 0]}

```
In[•]:= solP0b = DSolve[Join[eqP0, {q1[0] == 0, q2[0] == 1, p1[0] == 0, p2[0] == 0}], {q1, q2, p1, p2}, t] //
  First
```

Out[•]= {p1 → Function[{t}, 0], q1 → Function[{t}, 0],
 p2 → Function[{t}, -Sin[t]], q2 → Function[{t}, Cos[t]]}

```
In[•]:= solP0c = DSolve[Join[eqP0, {q1[0] == 0, q2[0] == 0, p1[0] == 1, p2[0] == 0}], {q1, q2, p1, p2}, t] //
  First
```

Out[•]= $\left\{p1 \to \text{Function}[\{t\}, 1], q1 \to \text{Function}\left[\{t\}, \frac{t}{2}\right], p2 \to \text{Function}[\{t\}, 0], q2 \to \text{Function}[\{t\}, 0]\right\}$

```
In[•]:= solP0d = DSolve[Join[eqP0, {q1[0] == 0, q2[0] == 0, p1[0] == 0, p2[0] == 1}], {q1, q2, p1, p2}, t] //
  First
```

Out[•]= {p1 → Function[{t}, 0], q1 → Function[{t}, 0],
 p2 → Function[{t}, Cos[t]], q2 → Function[{t}, Sin[t]]}

Then the fundamental matrix in zero approximation $X_0(t)$ is

```
In[•]:= X0[t_] = {{q1[t], q2[t], p1[t], p2[t]} /. solP0a, {q1[t], q2[t], p1[t], p2[t]} /. solP0b,
    {q1[t], q2[t], p1[t], p2[t]} /. solP0c, {q1[t], q2[t], p1[t], p2[t]} /. solP0d} // Transpose;
X0[t] // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & \frac{t}{2} & 0 \\ 0 & \text{Cos}[t] & 0 & \text{Sin}[t] \\ 0 & 0 & 1 & 0 \\ 0 & -\text{Sin}[t] & 0 & \text{Cos}[t] \end{pmatrix}$$

Now we compute the function $Y_1(t)$.

```
In[•]:= Y[0, t_] = {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};

In[•]:= JJ = {{0, 0, 1, 0}, {0, 0, 0, 1}, {-1, 0, 0, 0}, {0, -1, 0, 0}};
        JH1[t_] = Coefficient[JJ.matH2t, ε, 1/2]

Out[•]= {{0, 0, 0, 0}, {-4 Cos[t], 0, 0, 0}, {0, -2 Sin[t], 0, 4 Cos[t]}, {-2 Sin[t], 0, 0, 0}}

In[•]:= Y[1, t_] = Integrate[X0[-τ].JH1[τ].X0[τ].Y[0, τ], {τ, 0, t}]
```

$$\text{Out[•]=}\ \left\{\left\{0, \frac{3}{8}(-2\,t\,\text{Cos}[2\,t]+\text{Sin}[2\,t]), 0, -\frac{t^2}{4}+\frac{3}{4}\text{Sin}[t]\,(-2\,t\,\text{Cos}[t]+\text{Sin}[t])\right\},\right.$$
$$\left\{-t-3\,\text{Cos}[t]\,\text{Sin}[t], 0, -\frac{t^2}{4}+\frac{3}{4}\text{Sin}[t]\,(-2\,t\,\text{Cos}[t]+\text{Sin}[t]), 0\right\},$$
$$\left.\left\{0, -3\,\text{Sin}[t]^2, 0, t+3\,\text{Cos}[t]\,\text{Sin}[t]\right\}, \left\{-3\,\text{Sin}[t]^2, 0, -\frac{3}{8}(-2\,t\,\text{Cos}[2\,t]+\text{Sin}[2\,t]), 0\right\}\right\}$$

We do similar calculations step by step to the sixth order.

Then the monodromy matrix is

```
In[•]:= XX =
  X0[2 π].(Y[0, 2 π] + Sqrt[ε] Y[1, 2 π] + ε Y[2, 2 π] + ε^(3/2) Y[3, 2 π] + ε^2 Y[4, 2 π] + ε^(5/2) Y[5, 2 π] +
      ε^3 Y[6, 2 π]) // Expand // Collect[#, ε, Simplify] &
```

$$\text{Out[•]=}\ \left\{\left\{1-3\pi^2\varepsilon+\frac{3}{64}\pi^2(57+16\pi^2)\varepsilon^2-\frac{3\pi^2(232\,545+160\,000\pi^2+8192\pi^4)\varepsilon^3}{327\,680},\right.\right.$$
$$-\frac{3\pi\sqrt{\varepsilon}}{2}+\frac{3}{256}\pi(53+64\pi^2)\varepsilon^{3/2}-\frac{9\pi(3795+10\,080\pi^2+1024\pi^4)\varepsilon^{5/2}}{81\,920},$$
$$\pi-\frac{1}{16}\pi(11+8\pi^2)\varepsilon+\left(\frac{317\pi}{512}+\frac{7\pi^3}{8}+\frac{3\pi^5}{40}\right)\varepsilon^2+\left(\frac{502\,941\pi}{1\,048\,576}-\frac{2351\pi^3}{2048}-\frac{27\pi^5}{128}-\frac{3\pi^7}{560}\right)\varepsilon^3,$$
$$\left.\pi^2\sqrt{\varepsilon}-\frac{1}{128}\pi^2(127+32\pi^2)\varepsilon^{3/2}+\left(\frac{29\,411\pi^2}{32\,768}+\frac{263\pi^4}{512}+\frac{\pi^6}{40}\right)\varepsilon^{5/2}\right\},$$
$$\left\{-2\pi\sqrt{\varepsilon}+\left(\frac{171\pi}{64}+2\pi^3\right)\varepsilon^{3/2}-\frac{3\pi(-32\,755+65\,280\pi^2+8192\pi^4)\varepsilon^{5/2}}{81\,920},\right.$$
$$1+\frac{3\pi^2\varepsilon}{2}-\frac{3}{64}\pi^2(23+8\pi^2)\varepsilon^2+\frac{3\pi^2(77\,505+72\,960\pi^2+4096\pi^4)\varepsilon^3}{327\,680},$$
$$-\pi^2\sqrt{\varepsilon}+\frac{1}{128}\pi^2(127+32\pi^2)\varepsilon^{3/2}-\frac{\pi^2(147\,055+84\,160\pi^2+4096\pi^4)\varepsilon^{5/2}}{163\,840},$$
$$\left.-\frac{1}{24}\pi(33+16\pi^2)\varepsilon+\frac{1}{960}\pi(-495+830\pi^2+96\pi^4)\varepsilon^2+\left(-\frac{496\,813\pi}{1\,179\,648}-\frac{5197\pi^3}{6144}-\frac{151\pi^5}{640}-\frac{\pi^7}{140}\right)\varepsilon^3\right\},$$
$$\left\{-9\pi\varepsilon+\left(\frac{141\pi}{32}+6\pi^3\right)\varepsilon^2+\left(\frac{59\,793\pi}{16\,384}-\frac{105\pi^3}{16}-\frac{9\pi^5}{10}\right)\varepsilon^3, \frac{9}{2}\pi^2\varepsilon^{3/2}-\frac{9}{256}\pi^2(79+32\pi^2)\varepsilon^{5/2},\right.$$
$$1-3\pi^2\varepsilon+\frac{3}{64}\pi^2(57+16\pi^2)\varepsilon^2-\frac{3\pi^2(232\,545+160\,000\pi^2+8192\pi^4)\varepsilon^3}{327\,680},$$
$$\left.2\pi\sqrt{\varepsilon}+\left(-\frac{171\pi}{64}-2\pi^3\right)\varepsilon^{3/2}+\frac{3\pi(-32\,755+65\,280\pi^2+8192\pi^4)\varepsilon^{5/2}}{81\,920}\right\},$$
$$\left\{-\frac{9}{2}\pi^2\varepsilon^{3/2}+\frac{9}{256}\pi^2(79+32\pi^2)\varepsilon^{5/2}, -\frac{9\pi\varepsilon}{4}+\frac{9}{256}\pi(9+32\pi^2)\varepsilon^2+\left(\frac{67\,781\pi}{147\,456}-\frac{693\pi^3}{512}-\frac{27\pi^5}{160}\right)\varepsilon^3,\right.$$
$$\frac{3\pi\sqrt{\varepsilon}}{2}-\frac{3}{256}\pi(53+64\pi^2)\varepsilon^{3/2}+\frac{9\pi(3795+10\,080\pi^2+1024\pi^4)\varepsilon^{5/2}}{81\,920},$$

$$1+\frac{3\pi^2\varepsilon}{2}-\frac{3}{64}\pi^2\left(23+8\pi^2\right)\varepsilon^2+\frac{3\pi^2\left(77505+72960\pi^2+4096\pi^4\right)\varepsilon^3}{327680}\Big\}\Big\}$$

The corresponding characteristic polynomial is

In[∘]:=
```
CharEq = (Det[XX - ρ IdentityMatrix[4]]) // Series[#, {ε, 0, 3}] & // Normal //
   Collect[#, {ε}, Simplify] & // Factor
```

Out[∘]=
$$\frac{1}{5120}(-1+\rho)^2\left(5120-10240\rho+15360\pi^2\varepsilon\rho-16320\pi^2\varepsilon^2\rho-3840\pi^4\varepsilon^2\rho+14535\pi^2\varepsilon^3\rho+8160\pi^4\varepsilon^3\rho+384\pi^6\varepsilon^3\rho+5120\rho^2\right)$$

It is clear that the system has two characteristic multipliers $\rho_{1,2}=1$ and another pair of characteristic multipliers are the roots of quadratic equation

In[∘]:=
```
CharEq1 = (CharEq / (-1 + ρ)^2) // Collect[#, {ρ, ε}, Simplify] &
```

Out[∘]=
$$1+\left(-2+3\pi^2\varepsilon-\frac{3}{16}\pi^2\left(17+4\pi^2\right)\varepsilon^2+\frac{3\pi^2\left(4845+2720\pi^2+128\pi^4\right)\varepsilon^3}{5120}\right)\rho+\rho^2$$

In[∘]:=
```
solρ = ρ → 1 + Sqrt[ε] ρ1 + ε ρ2 + ε^(3/2) ρ3 + ε^2 ρ4 + ε^(5/2) ρ5 + ε^3 ρ6;
```

In[∘]:=
```
ρρ1 = (ρ /. Solve[CharEq1 == 0, ρ][[1]]) /. solρ // Series[#, {ε, 0, 3}] & // Normal //
   Collect[#, {ε}, Simplify[#, ε > 0] &] &
```

Out[∘]=
$$1-i\sqrt{3}\pi\sqrt{\varepsilon}+\frac{1}{32}i\pi\left(48i\pi+17\sqrt{3}\sqrt{\varepsilon}+16\sqrt{3}\pi^2\sqrt{\varepsilon}\right)\varepsilon-\frac{i\pi\left(2040i\pi+480i\pi^3+425\sqrt{3}\sqrt{\varepsilon}+1020\sqrt{3}\pi^2\sqrt{\varepsilon}+96\sqrt{3}\pi^4\sqrt{\varepsilon}\right)\varepsilon^2}{1280}-\frac{3\pi^2\left(4845+2720\pi^2+128\pi^4\right)\varepsilon^3}{10240}$$

In[∘]:=
```
ρρ2 = (ρ /. Solve[CharEq1 == 0, ρ][[2]]) /. solρ // Series[#, {ε, 0, 3}] & // Normal //
   Collect[#, {ε}, Simplify[#, ε > 0] &] &
```

Out[∘]=
$$1+i\sqrt{3}\pi\sqrt{\varepsilon}-\frac{1}{32}\pi\left(48\pi+17i\sqrt{3}\sqrt{\varepsilon}+16i\sqrt{3}\pi^2\sqrt{\varepsilon}\right)\varepsilon+\frac{\pi\left(2040\pi+480\pi^3+425i\sqrt{3}\sqrt{\varepsilon}+1020i\sqrt{3}\pi^2\sqrt{\varepsilon}+96i\sqrt{3}\pi^4\sqrt{\varepsilon}\right)\varepsilon^2}{1280}-\frac{3\pi^2\left(4845+2720\pi^2+128\pi^4\right)\varepsilon^3}{10240}$$

The corresponding characteristic exponents are given by

In[∘]:=
```
1/(2 π) Log[ρρ1] // Series[#, {ε, 0, 3}] & // Normal // Collect[#, {ε}, Simplify] &
```

Out[∘]=
$$-\frac{1}{2}i\sqrt{3}\sqrt{\varepsilon}+\frac{17}{64}i\sqrt{3}\varepsilon^{3/2}-\frac{85}{512}i\sqrt{3}\varepsilon^{5/2}$$

In[∘]:=
```
1/(2 π) Log[ρρ2] // Series[#, {ε, 0, 3}] & // Normal // Collect[#, {ε}, Simplify] &
```

Out[∘]=
$$\frac{1}{2}i\sqrt{3}\sqrt{\varepsilon}-\frac{17}{64}i\sqrt{3}\varepsilon^{3/2}+\frac{85}{512}i\sqrt{3}\varepsilon^{5/2}$$

Conclusion

In case of $\varepsilon=0$ the system has four characteristic exponents $\lambda_{1,2}=0$ and $\lambda_{3,4}=\pm i$ and its equilibrium state is unstable.

But for $\varepsilon>0$ the system has two pairs of purely imaginary complex conjugate characteristic exponents what means stability of the dynamic equilibrium described by the periodic solution of the equations of motion we have found at the beginning of this talk.

$$\lambda_{1,2}=\pm i\frac{\sqrt{3\varepsilon}}{2}\left(1-\frac{17\varepsilon}{32}+\frac{85\varepsilon^2}{256}\right);\quad \lambda_{3,4}=\pm i.$$

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