Dynamics of quantum states generated by the nonlinear Schrödinger equation

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Plan

1. Cauchy problem for NSE: global existense or gradient blow up.

2. Gradient blow up, self-focucing and destruction of the quantum state.

3. Continuation of the dynamics with transition into the space of mixed quantum states.

4. The extended dynamics as the partial trace of the nonlinear dynamics of pure states in the extended Hilbert space.

1. Cauchy problem for the nonlinear Schrodinger equation

Cauchy problem for the nonlinear Schrodinger equation on the segment:

$$i\frac{du}{dt} = \mathbf{L}u(t) \equiv -\boldsymbol{\Delta}u(t) - |u(t)|^p u(t), \ \ t \in (0, T); \qquad (1.1)$$

$$u(+0) = u_0; \quad u_0 \in H \equiv L_2(D). \qquad (1.2)$$

where $u_0 \in H = L_2(D), \ T \in (0, +\infty], \qquad p \geqslant 0.$
u is unknown map $[0, T) \to H$ which satisfies (1.1) and (1.2) (see Definition below).

$\boldsymbol{\Delta}$ is Laplace operator on the domain D.

$D = (-\pi, \pi) \subset \mathbb{R}.$

Solution of nonlinear Cauchy problem

$\boldsymbol{\Delta}$ is Laplace-Dirichlet operator.
$\mathbb{D}(\boldsymbol{\Delta}) = \{u \in W_2^2(-\pi, \pi) : \ u(-\pi) = 0 = u(\pi)\}$.
$H^l = \mathbb{D}((-\boldsymbol{\Delta})^{l/2})$, $l \in 0, 1, \ldots$.

Definition

The function u is called H^l-solution for Cauchy pronlem (1.1), (1.2) with some $l \in \mathbf{N}$ if $u \in C([0, T), H^l)$ and

$$u(t) = e^{-it\boldsymbol{\Delta}} u_0 - i \int_0^t e^{-i(t-s)\boldsymbol{\Delta}} [|u(s)|^p u(s)] ds, \ t \in [0, T). \quad (1.3)$$

Let $N(u) = \|u\|_H^2, \ u \in H$;

$$E(u) = \int_D [\frac{1}{2} |\nabla u\|^2 - \frac{1}{p+2} |u|^{p+2}] dx, \ u \in H^1.$$

Local and global solvability of nonlinear Cauchy problem

Theorem 1.1.

Let $D = (-\pi, \pi)$, and $p \geqslant 0$. Then the following statement holds: $\forall \quad \rho > 0 \quad \exists \quad T_* = T_*(\rho) > 0$ *such that if* $u_0 \in H^1$ *and* $\|u_0\|_{H^1} \leqslant \rho$ *then the Cauchy problem (1.1), (1.2) has the unique* H^1*-solution* $u_{u_0} = \mathcal{R}(u_0) \in C([0, T_*], H^1)$.

Theorem 1.2.

If $0 \leqslant p < 4$ *then for any* $u_0 \in H^1$ *Cauchy problem (1.1), (1.2) has the unique* H^1*-solution on the semiaxe* R_+.

Then one-parametric family $\mathbf{V}_t,\ t \in \mathbb{R}$ *of mappings* $H^1 \to H^1$ *acting by the rule* $\mathbf{V}_t u_0 = u_{u_0}(t),\ t \in \mathbb{R}$ *is the one-parametric group of continuous nonlinear mappings* $H^1 \to H^1$. *In addition,* $\|\mathbf{V}_t u_0\|_H = \|u_0\|_H,\ E(\mathbf{V}_t u_0) = E(u_0),\ t \in \mathbb{R}$ *for all* $u_0 \in H^1$.

Gradient blow up phenomenon

Theorem 1.3.
Let $p \geqslant 4$, *and* $u_0 \in H^1$ *satisfy the condition* $E(u_0) < 0$. *Then there is a number* $T^* \geqslant T_*$ *(see Theorem 1.1) such that supremum* T_1 *of the* H^1*-solution existence interval of the Cauchy problem (1.1), (1.2) satisfies the inequalities* $T_* \leqslant T_1 \leqslant T^*$.
$\|u(t)\|_H = \|u_0\|_H, \; E(u(t)) = E(u_0), \; \forall \; t \in [0, T_1)$.
Moreover, the limit equality holds:

$$\lim_{t \to T_1 - 0} \|u(t)\|_{H^1} = +\infty.$$

Regularizatio of NSE

Unboundedness of level surfaces of the energy functional $E(u)$ in the space H^1 is the reason of the gradient catastrophe for large p. The regularization of NSE (1.1) is the one-parameter family of the nonlinear Schrödinger equations such that its energy functional has the bounded energy level surfaces.
For example,

$$i\frac{d}{dt}u = \mathbf{L}_\epsilon u \equiv \mathbf{\Delta} u + V_\epsilon(|u|)u,\ t > 0, \quad \epsilon \in (0,1),\ \epsilon \to 0, \quad (1.4)$$

$$V_\epsilon(|u|) = \frac{1}{1+\epsilon^2|u|^{2p+4}}|u|^{p+2},\ \epsilon \in (0,1).$$

The regularized energy functional for every $\epsilon \in (0,1)$ has the form

$$E_\epsilon(u) = \int\limits_{-\pi}^{\pi}[\frac{1}{2}|\nabla u|^2 - \frac{1}{\epsilon(p+2)}\,\mathrm{arctg}(\epsilon|u|^{p+2})]dx,\ u \in H^1.$$

Solution of regularized problem

Let $\epsilon \in (0,1)$, $T \in (0,+\infty]$, and $l \in \mathbb{N}$. A function $u_\epsilon \in C([0,T), H^l)$ is called the H^l-solution of the Cauchy problem (1.2), (1.4) on the segment $[0,T)$ if it satisfies the equality

$$u_\epsilon(t) = e^{-i\mathbf{\Delta} t}u_0 + \int\limits_0^t e^{-i\mathbf{\Delta}(t-s)}V_\epsilon(|u_\epsilon(s)|)u_\epsilon(s)ds, \;\; t \in [0,T).$$

Theorem 1.5. *Let $\epsilon > 0$, $p \geqslant 0$. Then for any $u_0 \in H^1$ the Cauchy problem (1.2), (1.4) on the interval $[0,+\infty)$ has the unique H^1-solution $u_\epsilon(t;u_0)$; moreover, functionals $N(u)$ and $E_\epsilon(u)$ take constant values on the range of a solution $u_\epsilon(t;u_0)$, $t \geqslant 0$.*

$$u_\epsilon(t;u_0) = \mathbf{W}_\epsilon(t)u_0, \;\; t \geqslant 0; \;\; u_0 \in H^1.$$

The continuous semigroup $\mathbf{W}_\epsilon(t)$, $t \geqslant 0$, of nonlinear mappings of the space H^1 has the unique continuous continuation onto the continuous semigroup of nonlinear mappings on the space H.

Convergence of the solutions of regularized problem

Theorem 1.6.
Let $u_0 \in H^1$. Let $T_1 \in (0, +\infty)$ be supremum of the interval, on which the H^1-solution $u(t; u_0)$, $t \in [0, T_1)$, of the Cauchy problem (1.1), (1.2) exists. Then for any $T \in (0, T_1)$ the directed family $\{u_\epsilon(t; u_0),\ t > 0, \}$ of solutions of the problems (1.2), (1.4) converges to the solution $u(t; u_0)$, $t \in [0, T_1)$ of the problem (1.1), (1.2) in the sense of the equality

$$\lim_{\epsilon \to 0} \sup_{t \in [0,T]} \|u_\epsilon(t; u_0) - u(t; u_0)\|_H = 0 \ \forall\ T \in [0, T_1).$$

If $p = 4$ and $T \geqslant T_1$ then there is no infinitesimal sequence $\{\epsilon_k\}$ such that the sequence $\{u_{\epsilon_k}\}$ converges in the space $C([0, T], H)$.

2. Blow up phenomenon for solution and the destruction of quantum state

Let $B(H)$ be the Banach algebra of bounded linear operators in the space H.

$T_1(H)$ be the Banach space of trace class operators.

$B^*(H)$ be the Banach space conjugated to the space $B(H)$.

$\Sigma(H) = S_1(B^*(H)) \bigcap (B^*(H))_+$ be the set of quantum states.

$\Sigma_p(H)$ be the set of pure states.

$$\rho_u : \; B(H) \to \mathbb{C}, \quad \langle \rho_u, \mathbf{A} \rangle = (u, \mathbf{A}u)_H, \; \mathbf{A} \in B(H).$$

$\Sigma_n(H) = S_1(\sigma_1(H)) \bigcap (T_1(H))_+$ be the set of normal states.
$\rho = \sum\limits_{k=1}^{\infty} p_k \rho_{u_k}, \; \{u_k\}$ *is ONB*.

Criteria of pure and normal state

Let
$\mathcal{P}(H)$ be the set of finite dimensional orthogonal projectors;
$\mathcal{P}_1(H)$ be the set of 1-dimensional orthogonal projectors;

Lemma 2.1.
The state ρ is pure iff $\sup\limits_{u \in \mathcal{P}_1(H)} \langle \rho, \mathbf{P}_u \rangle = 1$.
The state ρ is normal iff $\sup\limits_{\mathbf{P} \in \mathcal{P}(H)} \langle \rho, \mathbf{P} \rangle = 1$.

Blow up phenomenon, self-focusing and state destruction

Definition

A solution $u(\cdot; u_0)$ of the Cauchy problem for Schrodinger equation admits

1) a gradient blow up phenomenon if there exists a number $T_1 \in (0, +\infty)$ such that $\lim\limits_{t \to T_1 - 0} \|u(t; u_0)\|_{H^1} = +\infty$;

2) a self-focusing phenomenon at the point $x_1 \in D$ if there exists a number $T_1 \in (0, +\infty)$ such that
$\lim\limits_{t \to T_1 - 0} \int\limits_D |x_1 - x|^2 |u(t, x; u_0)|^2 dx = 0,$

3) a pure state destruction if there are numbers $T_1 \in (0, +\infty)$ and a sequence $\{t_k\}$ such that $t_k \to T_1 - 0$, and a sequence $\{u(t_k; u_0)\}$ weakly converges to $u_* \in H$ such that $\|u_*\| < \|u_0\|$.

4) a normal state destruction if there are numbers $T_1 \in (0, +\infty)$ and a sequence $\{t_k\}$ such that $t_k \to T_1 - 0$ and the ineqaulity $\sup\limits_{\mathbf{P} \in \mathcal{P}(H)} [\lim\limits_{k \to \infty} \langle \rho_{u(t_k, u_0)}, \mathbf{P} \rangle] < 1$ holds.

Blow up phenomenon, self-focusing and state destruction

Point out the correlations between phenomena of the gradient blow up, the destruction of a pure state and the solution self-focusing.

Theorem 2.1. *Let* $T_1 \in (0, +\infty)$ *and* $u(t; u_0)$, $t \in [0, T_1)$ *be an* H^1*-solution of the Cauchy problem for Schrodinger equation (1.1), (1.2).*
Then the following implications are valid $d) \Rightarrow c) \Rightarrow b) \Rightarrow a)$.
Here conditions $a)$, $b)$ *and* $c)$ *mean the following:*
a) a solution admits the gradient blow up for $t \to T_1 - 0$;
b) a solution admits the destruction of pure state for $t \to T_1 - 0$;
c) a solution admits the destruction of normal state for $t \to T_1 - 0$;
d) a solution admits the self-focusing phenomenon for $t \to T_1 - 0$.
Moreover, $a) \Rightarrow b)$ *for Cauchy problem (1.1), (1.2) for NSE with* $p = 4$.

3. Regularized dynamics in the set of states for NSE

Let $p \geqslant 0$, $\epsilon > 0$. The group $\mathbf{T}_\epsilon$ acts on an element $\rho_{u_0} \in \Sigma_p(H)$ by the rule

$$\mathbf{T}_\epsilon(t)\rho_{u_0} = \rho_{\mathbf{W}_\epsilon(t)u_0}, \ t \in \mathbb{R}, \ \rho_{u_0} \in \Sigma_p(H).$$

We study the limit points of the directed family in weak-* topology of the space $(B(H))^*$

$$\mathbf{T}_\epsilon \rho_{u_0} = \rho_\epsilon(t, \rho_{u_0}), \ \epsilon \to 0.$$

Random dynamics in the set of quantum states

Let $\mathcal{A}^*$ be the σ-algebra of subsets generated by the family of functionals $\{\Phi_{\mathbf{A}} : \ \rho \to \rho(\mathbf{A}), \ \mathbf{A} \in B(H)\}$ on the set $\Sigma(H)$.

Let $W_0(0,1)$ be the set of nonnegative finite additive measures on the measurable space $(\Omega, \mathcal{F}) = ((0,1), 2^{(0,1)})$ concentrated in an arbitrary punctured right half-neighborhood of the point 0 and normalized by the equality $\nu((0,1)) = 1$. Here $2^{(0,1)}$ is the σ-algebra of all subsets of the interval (0, 1).

The solutions of regularized Cauchy problems (1.2), (1.4) and the measure $\nu \in W_0(\Omega)$ on the measurable space $(\Omega, \mathcal{F})$ define the random process with values in the set $\Sigma_p(H)$.

$$(\Omega, \mathcal{F}, \nu) \times \mathbb{R} \quad \to \quad (\Sigma_p(H), \mathcal{A}^*)$$

$$\Omega \times \mathbb{R} \to \Sigma_p(H); \qquad (\epsilon, t) \to \rho_{\mathbf{W}_\epsilon(t) u_0}$$

The continuation of solution by the random process

Theorem 3.1. *Let* $\nu \in W_0(0,1)$, $u_0 \in H^1$, *and* $[0, T_1)$ *be the existence interval of the* H^1*-solution for Cauchy problem (1.1), (1.2). Then the mean value of random process* $\rho_{u_\epsilon(t;u_0)}$, $t \in \mathbb{R}_+$, *defines the one-parameter family of quantum states*

$$\mathcal{T}^\nu(t)\rho_{u_0} = \rho^\nu(t, \rho_{u_0}); \quad \rho^\nu(t, \rho_{u_0}) = \int\limits_\Omega \rho_{u_\epsilon(t,u_0)} d\nu(\epsilon),\ t \in \mathbb{R}_+$$

which has the following properties

i) $\rho^\nu(t, \rho_{u_0}) = \rho_{u(t;u_0)}\ \forall\ t \in [0, T_1)$;

ii) $\forall\ t \geqslant 0\ \rho^\nu(t, \rho_{u_0}) \in \Sigma(H)$ *is the limit point in weak-* topology of the directed family of regularized states* $\{\mathbf{T}_\epsilon(t)\rho_{u_0},\ \epsilon \to 0\}$;

iii) $\rho^\nu(T_1, \rho_{u_0}) \notin \Sigma_n(H)$ *if* $p = 4,\ T_1 < +\infty$.

The continued solution as the partial trace of the pure states nonlinear dynamics in the extended Hilbert space.

The one-parametric family of dynamical mappings $\mathcal{T}^{\nu}(t)$, $t \geqslant 0$, can be presented as the partial trace of one-parametric semigroup of nonlinear mappings of pure states set in the extended Hilbert space.

$$\mathcal{H} = L_2((0,1), 2^{(0,1)}, \nu, H).$$

$$U_0(\epsilon) = u_0,\ \epsilon \in (0,1).$$

$$\mathcal{U}(t)U_0(\epsilon) = u_{\epsilon}(t, U_0(\epsilon)),\ \ t \geqslant 0,\ \ \epsilon \in (0,1).$$

$\mathcal{U}(t) :\ \mathcal{H} \to \mathcal{H}$ is the one-parametric group of nonlinear mappings.

Theorem 3.2. *Let $\nu \in W_0(0,1)$, $u_0 \in H$. Then $\mathcal{T}^{\nu}(t)\rho_{u_0}$ is the partial trace of pure vector state $\rho_{\mathcal{U}(t)U_0} \in \Sigma_p(\mathcal{H})$ which is defined as the restriction of the state $\rho_{\mathcal{U}(t)U_0}$ onto C^* subalgebra* $\mathcal{A}_H = B(H) \otimes \mathbf{I}_E :\ \ \mathbf{A} \otimes \mathbf{I}_E U(\epsilon) = \mathbf{A} U(\epsilon),\ \epsilon \in (0,1),\ \ \mathbf{A} \in B(H).$

$$\langle \mathcal{T}^{\nu}(t)\rho_{u_0}, \mathbf{A} \rangle = (\mathcal{U}(t)U_0, (\mathbf{A} \otimes \mathbf{I}_E)\mathcal{U}(t)U_0),\ \ \mathbf{A} \in B(H).$$

The continuation of solution by the random process

The solution of Cauchy problem (1.1), (1.2) is continued on the semiaxe $[0, +\infty)$

by the random process $\mathcal{T}\rho_{u_0} : \ \Omega \times \mathbb{R}_+ \to \Sigma_p(H)$;
$\Leftrightarrow$
by the one-parametric family of quantum states $\mathcal{T}^{\nu}(t)\rho_{u_0}$, $t \geqslant 0$;
$\Leftrightarrow$
by the partial trace of the pure states nonlinear dynamics in the extended Hilbert space.

One-parametric family $\mathcal{T}^{\nu}(t)$, $t \geqslant 0$, is not a semigroup.

The sequence of iterations $\{\mathcal{S}_n(t) = (\mathcal{T}^{\nu}(\frac{t}{n}))^n, \ t \geqslant 0\}$ can be approximation of some averaged semigroup (Volovich, Sakbaev 2018).

Thank you for attention!!!

Conclusions

The following questions are studied:

Regularization of Cauchy problem for NSE.

The set of limit points of directed set of regularizing problems.

The relationship between the phenomena of gradient blow up, self-focusing and destruction of quantum state.

The extension of one-parametric family of dynamical mappings on the quantum state set through the moment of blow up.

Thank you for attention