Integrability of quantum and classical dynamical systems

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Plan

- System of classical harmonic oscillators
- Quantum dynamical systems
- Classical dynamical systems. PDE.
- Wave operators. Classical and quantum field theory
- Remarks on monoidal and triangulated categories

Main results

- **Any quantum dynamical system is completely integrable.**
- **Moreover, it is unitary equivalent to a set of classical noninteracting harmonic oscillators**
- **Explicit integrability wide classes of classical and quantum multidimensional PDE by using wave operators.**

Harmonic oscillators

Definition. Complexified system of classical harmonic oscillators

$$(L^2(X,\mu),\, \omega,\ V_t = e^{-it\omega})$$

,
$\omega : X \to \mathbb{R}$. If $f \in L^2(X,\mu)$ then $V_t f(x) = e^{-it\omega_x} f(x)$.

Equations of motion. $\varphi_x = e^{-it\omega_x} f(x) = q_x + ip_x$

$$i\dot{\varphi}_x = \omega_x \varphi_x$$

$$\dot{q}_x = \omega_x p_x, \quad \dot{p}_x = -\omega_x q_x$$

Harmonic oscillators

Example.

$$X = \mathbb{R},\ \mu(x) = \sum_{j=1}^{N} c_j \delta(x - x_j)$$

Then $L^2(X, \mu) = \mathbb{C}^N$ and for $f = (f_!, ..., f_N) \in \mathbb{C}^N$ one has

$$V_t e_j = e^{-it\omega_j} e_j$$

Quantum dynamical systems

Quantum dynamical system: $(\mathcal{H},\ U_t)$.

Schrödinger equation ($\psi = U_t\,\psi_0 = e^{-itH}\,\psi_0$)

$$i\frac{\partial\psi}{\partial t} = H\psi$$

Integrability of Quantum Dynamical Systems

Theorem. *Let* $(\mathcal{H},\ U_t)$ *be q quantum dynamical systems. Then there exists a complexified system of classical harmonic oscillators* $(L^2(X,\mu),\ \omega,\ V_t = e^{-it\omega})$ *and a unitary map* $W: \quad \mathcal{H} \to L^2(X,\mu)$ such that

$$U_t = W^* \, V_t \, W$$

Also one has on an appropriate domain

$$H = W^* M_\omega W$$

Integrability in category of Hilbert spaces.

Linear and non-linear Schrodinger equations

Linear Schrodinger equation includes "nonlinear Schrodinger equations" if one uses the second quantization.
Let $\mathcal{H} = \mathcal{F}(L^2(\mathbb{R}^n))$ Fock space and
$[\psi(x), \hat{\psi}^*(y)] = \delta(x-y)$, $x, y \in \mathbb{R}^n$.

$$H = -\int_{\mathbb{R}^n} \hat{\psi}^*(x)\Delta\hat{\psi}(x)\, dx + \frac{1}{2}\int \hat{\psi}^*(x)\hat{\psi}^*(y)v(x-y)\hat{\psi}(x)\hat{\psi}(y)$$

then $\hat{\psi}(x,t) = e^{itH}\hat{\psi}(x)e^{-itH}$ satisfies the nonlinear Schrodinger equation

$$i\partial_t\hat{\psi}(x,t) = -\Delta\hat{\psi}(x,t) + \int \hat{\psi}^*(y)v(x-y)\hat{\psi}(x)\hat{\psi}(y)dy$$

$$\hat{\psi}(x,t) = W^* e^{it\omega} W \hat{\psi}(x) W^* e^{-it\omega} W$$

Historical remarks

Integrable systems: Newton, Euler, Lagrange,..., Poincare,...
Liouvilles theorem: *If a Hamiltonian system with n degrees of freedom has n independent integrals in involution, then itcan be integrated in quadratures. Canonical transform*.

KdW, Gardner, Green, Kruskal, Miura,... Zakharov, Novikov,...

Quantum integrable systems: Bethe, Yang, Baxter,...
Faddeev, Aref'eva-Korepin,...

Kozlov, Treschev,...

Vladimirov-I.V., ...
Accardi, Khrennikov, M. Ivanov

Finite dimensional case

We first consider the case of a finite-dimensional Hilbert space

$$\mathcal{H} = \mathbb{C}^n$$

Theorem 1. The Schrödinger equation $i\dot{\psi} = H\psi$, where H is a Hermitian operator in $\mathbb{C}^n$, $\psi = \psi(t) \in \mathbb{C}^n$, regarded as a classical Hamiltonian system, with a symplectic structure obtained by decomplexification the Hilbert space, has n independent integrals in involution.

Proof of Theorem 1;1/2

Let ω_j, $j = 1, ..., n$ be the eigenvalues of the operator H. By diagonalizing the matrix H

$$i\dot{\varphi}_j = \omega_j \varphi_j, \quad \varphi_j = \varphi_j(t) \in \mathbb{C},\ j = 1, ..., n.$$

Passing to the real and imaginary parts of the function $\varphi_j(t) = (q_j(t) + ip_j(t))/\sqrt{2}$,

$$\dot{q}_j = \omega_j p_j,\ \dot{p}_j = -\omega_j q_j.$$

with the Hamiltonian

$$H_{osc} = \sum_{j=1}^{n} \frac{1}{2}\omega_j (p_j^2 + q_j^2) = (\psi, H\psi).$$

Proof of Theorem 1; 2/2

Define

$$I_j(\varphi) = |\varphi_j|^2 = \frac{1}{2}(p_j^2 + q_j^2), \; j = 1, ..., n.$$

The functions I_j are integrals of motion, independent and in involution, $\{I_j, I_m\} = 0, \; j, m = 1, ..., n.$. The corresponding level manifold has the form of an n-dimensional torus.
Note that the Hamiltonian is a linear combination of the integrals of motion:

$$H_{osc} = \sum_{j=1}^{n} \omega_j I_j.$$

V.V. Kozlov (nondegenerate spectrum)

Complete integrability of quantum dynamical systems

The Schrödinger equation is completely integrable in the sense that its solutions are unitary equivalent to the complexified solutions of the Hamilton equations for a family of classical harmonic non-interacting oscillators.
The Schrödinger equation after the unitary transformation is rewritten in the form of equations for the family of classical harmonic oscillators. This is a consequence of the spectral theorem.

Theorem 2.

Let $\mathcal{H}$ be a separable Hilbert space and H a self-adjoint operator with a dense domain $D(H) \subset \mathcal{H}$. Then the Schrödinger equation

$$i \frac{\partial}{\partial t} \psi(t) = H\psi(t), \;\; \psi(t) \in D(H), \;\; t \in \mathbb{R}$$

completely integrable in the sense that this equation is unitary equivalent to the complexified system of equations for a family of classical non-interacting harmonic oscillators.
There exists a set of non-trivial integrals of motion for the arbitrary Schrödinger equation.

Proof of Theorem 2; 1/2

The solution of the Cauchy problem for the Schrödinger equation

$$i\frac{\partial}{\partial t}\psi(t) = H\psi(t), \;\; \psi(0) = \psi_0 \in D(H)$$

by the Stone theorem has the form $\psi(t) = U_t\psi_0$, where $U_t = e^{-itH}$ is the group of unitary operators, $t \in \mathbb{R}$.

Proof of Theorem 2; 2/2

Then, by the spectral theorem there exists a measurable space (X, Σ) with σ-finite measure μ, and a measurable finite a.e. function $\omega : X \to \mathbb{R}$ such that there is a unitary transformation $W : \mathcal{H} \to L^2(X, \mu)$ such that $U_t = W^* U_t^{(0)} W$ where $U_t^{(0)} = e^{-itM_\omega}$, where M_ω is the operator of multiplication by the function ω. On the corresponding domain one has

$$H = W^* M_\omega W$$

.

Proof of Theorem 2; 2/2

The initial Schrödinger equation in $\mathcal{H}$ goes over under the unitary transformation of W into the Schrödinger equation in $L^2(X, \mu)$ of the form

$$i\frac{\partial}{\partial t}\varphi_x(t) = \omega_x \varphi_x(t),\ x \in X$$

Passing to the real and imaginary parts of the function

$$\varphi_x(t) = \frac{1}{\sqrt{2}}(q_x(t) + ip_x(t)),$$

this Schrodinger equation is rewritten in the form of equations of the family of classical harmonic oscillators

$$\dot{q}_x = \omega_x p_x,\ \dot{p}_x = -\omega_x q_x,\ x \in X.$$

Hamiltonian

These equations can be obtained from the Hamiltonian

$$H_{osc} = \int_X H_x d\mu, \quad H_x = \frac{1}{2}\omega_x(p_x^2 + q_x^2),$$

by using the relation

$$\delta H_{osc} = \int_X \delta H_x d\mu = \int_X (\dot{q}_x \delta p_x - \dot{p}_x \delta q_x) d\mu.$$

Note also that one has $H_{osc} = (\omega\varphi, \varphi)_{L^2} = (H\psi, \psi)_{\mathcal{H}}$.

Integrals of motion.

Lemma 1. Let $I : L^2(X) \to \mathbb{R}$ **be an integral of motion for the dynamics** $U_t^{(0)}$, $I(U_t^{(0)}\varphi) = I(\varphi)$, $\varphi \in L^2(X)$. **Then** $J : \mathcal{H} \to \mathbb{R}$, **where** $J(\psi) = I(W\psi), \psi \in \mathcal{H}$, **is the integral of motion for dynamics** U_t, $J(U_t\psi) = J(\psi)$.

Integrals of motion for the dynamics $U_t^{(0)}$**:**

$$I_\gamma = I_\gamma(\varphi) = \int_X \gamma_x |\varphi_x|^2 d\mu = \int_X \gamma_x \frac{1}{2}(p_x^2 + q_x^2) d\mu,$$

for any $\gamma \in L^\infty(X)$**. Then, by Lemma 1, one gets an integral of motion** J_γ **for dynamics** U_t**. The theorem is proved.**

Where is quantum chaos?

An arbitrary quantum dynamics is unitary equivalent to a system of classical harmonic oscillators. So, what about quantum chaos?

By the Liouville-Arnold theorem the dynamics the n-harmonic oscillators is reduced to the motion on the n-dimensional torus T^n with equations of motions

$$\dot{z} = \lambda_i, \; i = 1, ...n$$

If λ_i are linearly independent with integer coefficients then the flow is ergodic, i.e. one has chaos.

Remark.

Consider an example when $X = \mathbb{R}$ and the Hilbert space is $L^2(\mathbb{R}, d\mu)$, where μ is a Borel measure on the line. Any measure μ on $\mathbb{R}$ admits a unique expansion in the sum of three measures $\mu = \mu_{pp} + \mu_{ac} + \mu_s$, where μ_{pp} is purely pointwise, μ_{ac} is absolutely continuous with respect to the Lebesgue measure and μ_s is continuous and singular with respect to Lebesgue measure, and we have respectively

$$L^2(\mathbb{R},\, d\mu) = L^2(\mathbb{R},\, d\mu_{pp}) \oplus L^2(\mathbb{R},\, d\mu_{ac}) \oplus L^2(\mathbb{R},\, d\mu_s).$$

A purely point measure has the form $\mu_{pp} = \sum c_j \delta_{s_j}$, where $c_j > 0$, the sum over j contains a finite or infinite number of terms, s_j real numbers and δ_{s_j} Dirac delta function. Then we have

$$\int |\varphi_x|^2 \, d\mu_{pp} = \sum_j c_j |\varphi_{s_j}|^2.$$

Examples

- Free particle. Fourier integral/series.
- Quantum oscillator. Coherent states.

On integrability of classical dynamical systems

The complete integrability of the Liouville equation in Koopman's approach to classical mechanics and in general any dynamical system is proved in a similar way. Let $(M, \Sigma, \alpha, \tau_t)$ be a dynamical system, where (M, Σ) measurable space with measure α and τ_t, $t \in \mathbb{R}$ group of measure-preserving transformations M. Then the Koopman transform defines a group of unitary operators U_t in $L^2(M, \alpha)$,

$$(U_t f)(m) = f(\tau_t(m)),\ f \in L^2(M, \alpha)$$

Repeating the proof of Theorem 2, we see that the group U_t is unitary equivalent to the family of harmonic oscillators and, in this sense, any dynamical system is **completely integrable in category of Hilbert spaces.**

Complexity

Compare the effectiveness (or complexity) of complete integrability in the above sense with the efficiency/complexity of constructing solutions of Hamiltonian systems that are completely integrable in the sense of Liouville.

Apparently, the change of variables in the Liouville theorem, which reduces the initial dynamics to action-angle variables on a level manifold, is an analogue of the unitary transformation in the spectral theorem, which reduces the original dynamics to a set of harmonic oscillators.

Open quantum systems?

Wave operators and integrability of classical dynamical systems

The inverse scattering method, based on the Lax representation, is used to prove the complete integrability of some nonlinear equations, usually on a straight line or on a plane.

Here we use of direct scattering theory methods to prove the complete integrability and the construction of integrals of motion for classical and quantum systems. This will provide a constructive example to the above general result on the complete integrability of arbitrary quantum and classical systems.

Wave operators and integrals of motion

Let $\phi(t)$ and $\phi_0(t)$ be a pair of groups of automorphisms of the phase space in the classical case or groups of unitary transformations in quantum case, $t \in \mathbb{R}$. Suppose there exist the limits, called wave operators,

Wave operators and integrals of motion

$$\lim_{t \to \pm\infty} \phi(-t)\phi_0(t) = \Omega_\pm$$

The wave operators have the properties of intertwining operators; on the appropriate domain, we have

$$\phi(t)\Omega_\pm = \Omega_\pm \phi_0(t)$$

Thus, the interacting dynamics $\phi(t)$ is reduced to "free" $\phi_0(t)$. The following form of Lemma 1 holds

Lemma 1a.

Let $I_0(z)$ be the integral of motion for the dynamics of $\phi_0(t)$, i.e. $I_0(\phi_0(t)z) = I_0(z)$, here z is a phase space point or a Hilbert space vector.
Then $I(z) = I_0(\Omega_\pm^{-1}z)$ is the integral of motion for the dynamics of $\phi(t)$.

Particle in a potential field. 1/2

Let $\Gamma = M \times M^{'}$ phase (symplectic) space, where $M = \mathbb{R}^n$ is the configuration space and $M^{'} = \mathbb{R}^n$ is the dual space of momenta. Hamiltonian

$$H(x, \xi) = \frac{1}{2}\xi^2 + V(x),$$

where $\xi \in M'$ and the function $V : M \to \mathbb{R}$ is bounded and the force $F(x) = - \bigtriangledown V(x)$ is locally Lipschitz. Then the solution $(x(t, y, \eta), \xi(t, y, \eta))$ of the equations of motion

$$\dot{x}(t) = \xi(t), \;\; \dot{\xi}(t) = F(x(t))$$

with initial data

$$x(0, y, \eta) = y, \;\; \xi(0, y, \eta) = \eta, \;\; y \in M, \, \eta \in M'$$

exists and is unique for all $t \in \mathbb{R}$.

Particle in a potential field. 2/2

Denote $\phi(t)(y,\eta) = (x(t,y,\eta), \xi(t,y,\eta))$. The free Hamiltonian is $H_0(x,\xi) = \xi^2/2$, the solution of the equations of motion has the form $\phi_0(t)(y,\eta) = (y + t\eta, \eta)$. Suppose

$$\lim_{|x|\to\infty} V(x) = 0, \int_0^\infty \sup_{|x|\geq r} |F(x)| dr < \infty$$

Then for any $(y,\eta) \in M \times M'$ there is a limit

$$\lim_{t\to\infty} \frac{x(t,y,\eta)}{t} = \xi_+(y,\eta),$$

moreover, if $\xi_+(y,\eta) \neq 0$, then there is a limit

$$\lim_{t\to\infty} \xi(t,y,\eta) = \xi_+(y,\eta),$$

and the inverse image of $\mathcal{D} = \xi_+^{-1}(M' \setminus \{0\})$ is the open set of all paths with positive energy unbounded for $t \to \infty$.

Proposition 1 (B.Simon,...)

Let the force satisfy the conditions

$$\int_0^\infty \sup_{|x|\geq r} |\partial_x^\alpha F(x)|(1+r^2)^{1/2} dr < \infty, \ |\alpha| = 0, 1.$$

Then there exists the limit

$$\lim_{t\to\infty} \phi(-t)\phi_0(t) = \Omega_+$$

uniformly on compact sets in $M \times (M' \setminus \{0\})$. The mapping $\Omega_+ : M \times (M' \setminus \{0\}) \to \mathcal{D}$ is symplectic, continuous and one-to-one. There are relations

$$H\,\Omega_+ = H_0, \ \ \phi(t)\,\Omega_+ = \Omega_+\,\phi_0(t).$$

Theorem 3.

The equations of motion of a particle in a potential field satisfying the above conditions determine an integrable system in the sense that the dynamics of $\phi(t)$ is symplectically equivalent to the free dynamics of $\phi_0(t)$ on the domain $\mathcal{D}$ in $2n$ - dimensional phase space, $\Omega_+^{-1}\phi(t)\,\Omega_+ = \phi_0(t)$. On $\mathcal{D}$ there are n independent integrals of motion in involution.

Wave operators and integrability in QM, 1/3

Consider the Schrödinger equation in $L^2(\mathbb{R}^n)$ of the form

$$i\frac{\partial\psi}{\partial t} = H\psi, \tag{1}$$

where $H = H_0 + V(x)$, $H_0 = -\triangle$ - Laplace operator and the potential V is short-range, i.e. it satisfies

$$|V(x)| \leq C(1+|x|)^{-\nu}, \;\; x \in \mathbb{R}^n, \;\; \nu > 1.$$

Then H defines a self-adjoint operator and there exist the wave operators $\Omega_\pm$, which are complete and diagonalize H, $H\Omega_\pm = \Omega_\pm H_0$.

Wave operators and integrability in QM, 2/3

Higher order integrals of motion.
If $\varphi = \varphi(t, x)$ is a smooth solution to the free Schrödinger equation $i\dot{\varphi} + \triangle\varphi = 0$ then its derivatives $\partial_x^\alpha \varphi$ also are solutions of this equation.
Hence, one can get higher integrals of motion just by replacing φ by $\partial_x^\alpha \varphi$ in the known integral $I_0(\varphi) = \int_{\mathbb{R}^n} |\varphi|^2 dx$.
One gets a set of integrals of motion: $I_\alpha = \int |\partial_x^\alpha \varphi|^2 dx$.
Now, according to Lemma 1a, we obtain integrals of motion $J_0, J_\alpha^\pm(\psi) = I_\alpha(\Omega_\pm^{-1}\psi)$ for the original Schrödinger equation with the potential $V(x)$.

Wave operators and Lippman-Schwinger equation

Constructing wave operators by solving the Lippman-Schwinger integral equation

$$\varphi(x,k) = e^{ikx} - \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|} V(y)\varphi(y,k)dy$$

$$\Omega_+ = K^{-1}F,$$

$$K^{-1}(x,k) = (2\pi)^{-3/2}\varphi(x,k)$$

Convergent series

Under the condition $\|V\|_R < 4\pi$, the series of perturbation theory for the solution $\varphi(x,k)$ converges:

$$\varphi(x,k) = e^{ikx} - \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|k||x-\xi|}}{|x-\xi|} V(\xi) e^{ik\xi} d\xi + \ldots$$

Rolnik potential. A measurable function V on $\mathbb{R}^3$ is called a Rolnik potential if

$$||V||_R = \int_{\mathbb{R}^3} \frac{|V(x)||V(y)|}{|x-y|^2} dxdy < \infty.$$

Wave operators and integrability for nonlinear partial differential equations, 1/2

Nonlinear Klein - Gordon equation

$$\ddot{u} - a^2 \triangle u + m^2 u + f(u) = 0, \ t \in \mathbb{R}, \ x \in \mathbb{R}^n,, \qquad (2)$$

where $a > 0, \ m \geq 0$.
Eq. (2) is a Hamiltonian system with the Hamiltonian

$$H = \int_{\mathbb{R}^n} (\frac{1}{2} p(x)^2 + \frac{1}{2} (\nabla u)^2 + \frac{1}{2} m^2 u^2 + V(u(x)) dx = H_0 + V \qquad (3)$$

where $V' = f$ and the the Poisson brackets are
$\{p(x), u(y)\} = \delta(x - y)$

Wave operators and integrability for nonlinear partial differential equations, 2/2

One can take $f(u) = \lambda |u|^2 u$, $\lambda \geq 0$ and $n = 3$. If the initial data $(u(0),\ \dot{u}_t(0)) \in H^{k+1}(\mathbb{R}^n) \times H^k(\mathbb{R}^n)$ then there exists a global solution $u(t)$ of Eq (2) with these initial data. Here $H^k(\mathbb{R}^n)$ is the Sobolev space. For any solution of (2) there exists a unique pair (v_+, v_-) of solutions for the free Klein-Gordon equation

$$\ddot{v} - a^2 \triangle v + m^2 v = 0, \tag{4}$$

$$\lim_{t \to \pm\infty} \|(u(t),\ \dot{u}(t)) - (v_\pm(t), \dot{v}_\pm(t))\| = 0.$$

Moreover, the correspondence $(u(0),\ \dot{u}(0)) \mapsto (v_\pm(0), \dot{v}_\pm(0))$ defines homeomorphisms (wave operators) $\Omega_\pm$ on $H^{k+1} \times H^k$.

Integrals of motion.

If $v = v(t, x)$ is a smooth solution of Eq. (4) then its partial derivatives $\partial_x^\alpha v$ is also the solution of Eq. (4). Therefore to get higher order integrals of motion from the energy integral

$$E(v) = \int (\dot{v}^2 + a^2(\nabla v)^2 + m^2 v^2)\, dx/2$$

one can just replace v by $\partial_x^\alpha v$. We get higher integrals of motion $I_\alpha(v) = E(\partial_x^\alpha v)$ for Eq. (4). One expects that if $v \in H^{|\alpha|}$ then $J_\alpha^\pm(u) = I_\alpha(\Omega_\pm u)$ will be integrals of motion for Eq. (2).

Categorical quantum dynamical system

The categorical quantum dynamical system is the triple $(\mathcal{T}, F, \mathcal{M})$, where $\mathcal{M}$ is a monoidal additive involutive category, and F is a functor from the category $\mathcal{T}$ to the category $\mathcal{M}$, $F : \mathcal{T} \to \mathcal{M}$, satisfying the following condition: if the functor F maps the object T to some object A, then $F : Hom(T, T) \to UHom(A, A)$. Moreover, $F(t)$ is a unitary morphism, $F(t) \in UHom(A, A)$ and $F(t + s) = F(t) \circ F(s)$.

Thus, the functor F defines a unitary representation of the abelian group T in monoidal additive categories with involution $\mathcal{M}$.

Integrals of motion

Integrals of motion in a monoidal additive involutive category. Let $\eta = (\mathcal{T}, F, \mathcal{M})$ be a categorical quantum dynamical system, F a functor from $\mathcal{T}$ to $\mathcal{M}$ such that $F(T) = A \in \mathcal{M}$ and $F : Hom(T, T) \to UHom(A, A)$. Further, let G be a functor from $\mathcal{T}$ to $\mathcal{M}$ such that $G(T) = A \in \mathcal{M}$ and $G : Hom(T, T) \to UHom(A, A)$. If the unitary morphisms $F(t)$ and $G(s)$ commute for all $t, s \in T$, i.e. $F(t) \circ G(s) = G(s) \circ F(t)$, then the functor G will be called *integral of motion* for the quantum dynamical system η.

Quantum quench

In physical works, quantum quench is usually called a **change of the state** of a quantum system under the influence of external perturbation. We will understand quantum quench in a generalized sense, when the **quantum system itself may change** under the influence of external perturbations and discuss in more detail a special type of quantum quenches, which will be called homological quantum quench or Pauli quench.

Homological quantum quench

Each physical system is associated with a Hilbert space. Consider the sequence of actions on the system, leading, possibly, not only to a change in the state of the system, but also to a change in the system itself, i.e. we have a sequence

$$\mathcal{H}_0 \xrightarrow{f_0} \mathcal{H}_1 \xrightarrow{f_1} \mathcal{H}_2 \xrightarrow{f_2} \ldots,$$

where $\mathcal{H}_n$ Hilbert spaces, and f_n continuous linear mappings (morphisms). Such chains of actions will be called quantum quenches.

Consider a special case of quantum quenches, when $f_n f_{n-1} = 0$, which we will call the sequence of Pauli quenches.

Triangulated categories

The category of Hilbert spaces *Hilb* is additive, but it is not Abelian. For any additive category $\mathcal{A}$, a triangulated category $\mathcal{K}(\mathcal{A})$ is constructed, a homotopy category of chain complexes over $\mathcal{A}$.
A triangulated category is the additive category $\mathcal{K}$ equipped with the additive shift functor [1]:
$\mathcal{K} \to \mathcal{K}$ and the class of so-called distinguished exact triangles, that is morphisms of $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$, satisfying some axioms.

$\mathcal{K} \to \mathcal{K}$ and the class of so-called distinguished exact triangles, that is morphisms of $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$, satisfying some axioms.

Triangulated categories and Quantum quenches

The monoidal triangulated category $\mathcal{K}(Hilb)$ of homotopy chain complexes is associated with the additive monoidal category of Hilbert spaces *Hilb*

The additive category $\mathcal{K}(Hilb)$ is a triangulated category. The shift functor is defined by graduation shifts, and the selected triangle is connected with the morphism of the complexes $f : K \to L$ and has the form $K \xrightarrow{f} L \to C(f) \to K[1]$, where the complex $C(f)$ is called the cone of the morphism f, we have $C(f)_n = K_{n+1} \oplus L_n$.

Summary

- Any quantum dynamical system is unitarily equivalent to the set of classical harmonic oscillators and in this sense is completely integrable in category of Hilbert spaces.
- Higher order integrals of motion are indicated for a number of quantum and classical dynamical systems by using wave operators.
- Triangulated categories describing quantum quenches are considered.